COMPENSATION OF SERVOMECHANISMS USING ROOT RELOCATION ZONE CONCEPTS

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UNITED STATES
NAVAL POSTGRADUATE SCHOOL

THESIS

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by

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Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
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ABSTRACT

The purpose of this thesis is to extend and amplify the compensator design concepts developed by E. R. Ross and T. C. Warren [8]. The design problem that motivates this investigation is: Given a linear feedback control system with unity feedback, design a simple cascaded lead or lag filter network taking into account the following constraints:

The open loop plant is fixed.

The system steady state error is to remain constant.

The dominant closed loop poles are specified.

The attack on this problem is divided into two parts:
(a) Derivation of compensator equations to solve for the numerical value of poles and zeros of the compensator network that will fulfill the three constraints. (b) An analysis of the zones of the s plane for which the compensator equations produce a certain type of solution. Generally these are the root relocation zones which indicate the range of points in the s plane at which a root can be located with a given multiplicity of compensator networks. Zones defined by various other constraints are also considered. This investigation was conducted at the United States Naval Postgraduate School under the direction of Dr. G. J. Thaler. Verification of results were carried out on an ESIAC computer provided under a research contract by the Office of Naval Research.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS

KG(s) The general form for a "transfer" function of the complex variable, s.

k is the associated gain constant and the form of $G(s)$ is

$$G(s) = \frac{\prod (s + Z_i)}{\prod (s + P_j)} = \frac{s^m + \ldots\ldots}{s^n + \ldots\ldots}$$

n is the system type number.

<u>Example (1):</u>

$$kG(s) = \frac{10\ (s + 2)}{(s + 0)(s + 1)} = \frac{10\ (s + 2)}{s(s + 1)}$$

In this example $k = 10$ and

$$G(s) = \frac{(s + 2)}{s(s + 1)} \qquad \text{and } n = 1.$$

K A symbol used in general to denote the error coefficient associated with $kG(s)$. (For systems other than unity feed back single loop systems, K is evaluated in the same manner but the interpretation is different)

It is normal to associate an alphabetic subscript with K corresponding to the system type number, n. i.e.,

K_v Type 1

K_a Type 2 etc.

In this paper other alphabetic subscripts will be used as described below. The subscript v will be used only to emphasize the fact that a system error coefficient is being considered.

In terms of $KG(s)$ as defined above;

$$K \quad \lim_{s \to 0} \quad s^n kG(s) \;=\; k \frac{\prod Z_i}{\prod P_j}$$

In example (1)

$$K = \frac{10\ (2)}{1} = 20$$

Z_i A zero of a function of a complex variable, $G(s)$.

$i = 1, 2, 3, \ldots\ldots$

$(s+Z_i)$ is a factor of $G(s)$, and $G(s) = 0$ when $s = -Z_i$. Therefore the singular point of $G(s)$ has the complex co-ordinates $-Z_i + j0$ if Z_i is a real number.

P_j A pole of a function of a complex variable, $G(s)$.

$j = 1, 2, 3, \ldots.$

$\frac{1}{(s + P_j)}$ is a factor of $G(s)$, and $G(s)$ is undefined when $s = -P_j$. Therefore the singular point of $G(s)$ has the complex co-ordinates $-P_j \pm j0$ if P_j is a real number.

If the real part of any given Z_i or P_j is a positive number, then the corresponding singular point of $G(s)$ lies in the left hand half of the s plane.

c "Closed loop" when appearing as the first of two subscripts. "Compensated" when appearing as a second subscript or alone with a Gain constant.

o "Open loop", when appearing as the first of two subscripts, or alone.

u "Uncompensated", in the sense that the compensator being designed for a system has not been applied. The u subscript does not imply instability.

L Denotes a parameter or function associated with a phase lead network when appearing as a subscript.

x Denotes a parameter or function associated with a phase lag network.

<u>Example (2)</u>:

$KG_o(s)$ A general open loop transfer function;

and with unity feedback

$KG_c(s) = \frac{kG_o(s)}{1 + kG_o(s)}$. The corresponding closed loop transfer function (unity feedback).

<u>Example (3)</u>:

$k_u G_{ou}(s)$: An open loop transfer function before a particular compensating device is applied.

$k_c G_{oc}(s)$: The open loop transfer function of the system after the compensator has been applied. If more than one system is being considered, different numerical subscripts will denote different systems.

Example (4):

$$G_L(s) = \frac{(s + Z_L)}{(s + P_L)}$$ Transfer function of a lead network.

(Gain factor not included.)

Example (5): Basic system with cascaded lead network.

$$k_c G_{oc}(s) = k_c G_o(s) G_L(s)$$

Where k_u may or may not be equal to k_c depending on the restrictions and specifications for the system.

$\alpha = \frac{Z}{P}$ The attenuation factor of a simple compensating network. A number of texts use the reciprocal of this ratio. (See Brown and Campbell, Principles of Servomechanisms.)

For a lead network $\alpha < 1$.

For a lag network $\alpha > 1$.

$-r$ A root of an equation (polynomial function) of a complex variable. $(s + r)$ is a factor of the equation and therefore the factor is zero when $s = -r$.

1. INTRODUCTION

The purpose of this thesis is to extend and amplify some of the concepts of compensator design developed in a Master's thesis by E. R. Ross and T. C. Warren [8] at the U. S. Naval Postgraduate School. The results of immediate interest will be published in the near future in a paper by Ross, Warren and Thaler. Specifically, the results of Chapter VI and VII of that thesis will be considered here. The two basic concepts will be referred to as "the compensator equations" and "root relocation zones."

The two concepts are but two parts of a single technique of design and analysis. The compensator equations permit numerical calculations of compensator parameters, but the equations in themselves do not answer questions about the validity of results. The concept of root relocation zones allows a determination in advance of actual computation the conditions which will result in valid and realizable numbers for the compensator poles and zeros. In many situations information contained in a graphical presentation of the root relocation zones permits a much more flexible approach to preliminary design than can be found the popular frequency response techniques.

Statement of the problem:

A very brief statement is: given a fixed plant, a specified low frequency gain, design a compensator to produce a desired pair of dominant complex roots. A more exact statement of the problem situation and the restrictions imposed will follow. A major portion of every recent book on control systems is devoted to the problem of compensation design. The first five books listed in the Bibliography bear out this point. Aseltine [7] reviews a number of recent articles on the design of compensation. In particular the passive lead and lag

network is discussed in considerable detail and therefore the derivation of the transfer functions, practical problems of application, etc. will not be repeated here. For example, see Thaler, Elements of Servo-mechanism Theory, Chapter 9, [2]. Assuming the preliminary information on compensation problems to be known, a more detailed statement of the topic of this thesis can be made.

A linear plant with an open loop transfer function of the following form is specified:

1.1 $$k_u G_{ou}(S) = \frac{k_u \prod (S+Z_i)}{S^n \prod (S+P_j)}$$

For which the error coefficient is defined as:

1.2 $$K_V = \lim_{S \to 0} S^n k_u G_{ou}(S)$$

The value of K_v is fixed, that is/; a steady state velocity error specification is to be met.

A point $-\Omega$ is chosen. $-\Omega$ is to be a root (Closed loop pole) of the compensated system. The motive for choosing this point is a desire to improve the transient response of the system.

<u>Restrictions</u>: The systems considered are linear. Unity feedback single loop systems are assumed in all derivations. <u>However</u> the results of thesis may be applied to <u>any</u> single loop of a linear control system; the concept of velocity error coefficient must be modified to that of "open loop zero frequence or D.C. gain", if there are transfer functions present in the feedback loop. The compensation applied to the system is restricted to networks or devices that are cascaded to

produce a compensated open loop system transfer function of the form:

1.3 $$k_c G_{oc}(S) = k_c G_{ou}(S) \frac{(S + Z_c)^{\eta}}{(S + P_c)^{\eta}}$$

Where Z_c is the compensator zero of multiplicity η and P_c is the compensator pole of multiplicity η. P_c AND Z_c are restricted to be real numbers. (There is no such restriction on the poles and zero of $G_{ou}(S)$ Further, since the compensator is to be interpreted to be a passive, RC network in most cases, it will be stipulated that only real, positive values of Z_c and P_c are of interest. This means that the singular points introduced into the system function must all be on the negative real axis.

<u>Solution and analysis</u>: By use of the compensator equations it is possible, under the restrictions listed above, to solve for exact values of Z_c and P_c. The analysis of the problem consists in dividing the s plane into zones in which it is possible to locate a root for specified conditions of compensator multiplicity and compensator pole zero ratio. An additional part of the analysis will be to consider the minimum pole zero spacing possible for a given choice of the point $-\eta$; and the limit of the attenuation factor as the multiplicity of compensator sections is increased for a chosen point .

The notation used is reasonably standard except for the matter of subscripts. A precedence for designating open and closed loop functions in this manner may be found in the article by Aseltine, [7]. It is hoped that the subscripts reduce confusion, rather than increase it.

The various root locus configurations developed were verified by

setting up examples on the ESIAC computer, One example of the ESIAC computer plot is included in Section 1 . The computer work was done in the form of a check out of the ESIAC which has been installed by the Office of Naval Research in connection with a research program to be conducted by Dr. G. J. Thaler.

2. DERIVATION OF A CONVENIENT FORM FOR THE GENERAL COMPENSATOR DESIGN EQUATIONS

One of the fundamental contributions of Ross and Warren [8] was the derivation of a set of relations between the given uncompensated system transfer function, the location of a desired pair of dominant roots and the pole/zero configuration of a passive compensating network necessary to produce the desired dominant roots.

Rather than review the Ross and Warren derivation; a similar argument will be carried out here with certain modifications that will result in a somewhat more convenient form for the "compensator equations".

A system, which will be called the uncompensated system, is specified by the open loop function

2.1 $$k_u G_{ou}(S) = \frac{k_u \left[(S+Z_1)\cdots(S+Z_i)\right]}{S^n\left[(S+P_1)\cdots(S+P_j)\right]}\ ;$$

and the system error coefficient is specified as

2.2 $$K_u = \lim_{S\to 0} S^n k_u G_{ou}(S) = k_u \frac{\prod^{i} Z_i}{\prod^{j} P_j}$$

It is further specified that the system error coefficient is to be invarient.

It is assumed that the theoretical (or actual) response of this system is undesirable. An indication of the undesirable performance being that the dominant roots of the system lie in the right hand half of the s plane or are otherwise badly located. The actual locations of the uncompensated system roots do not enter into the derivation which follows. Reference is made to the roots only as a possible motive

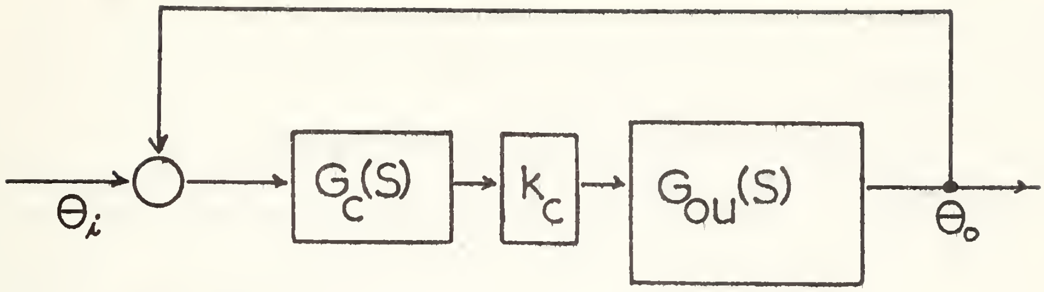


FIGURE 2-1 BLOCK DIAGRAM OF COMPENSATED SYSTEM

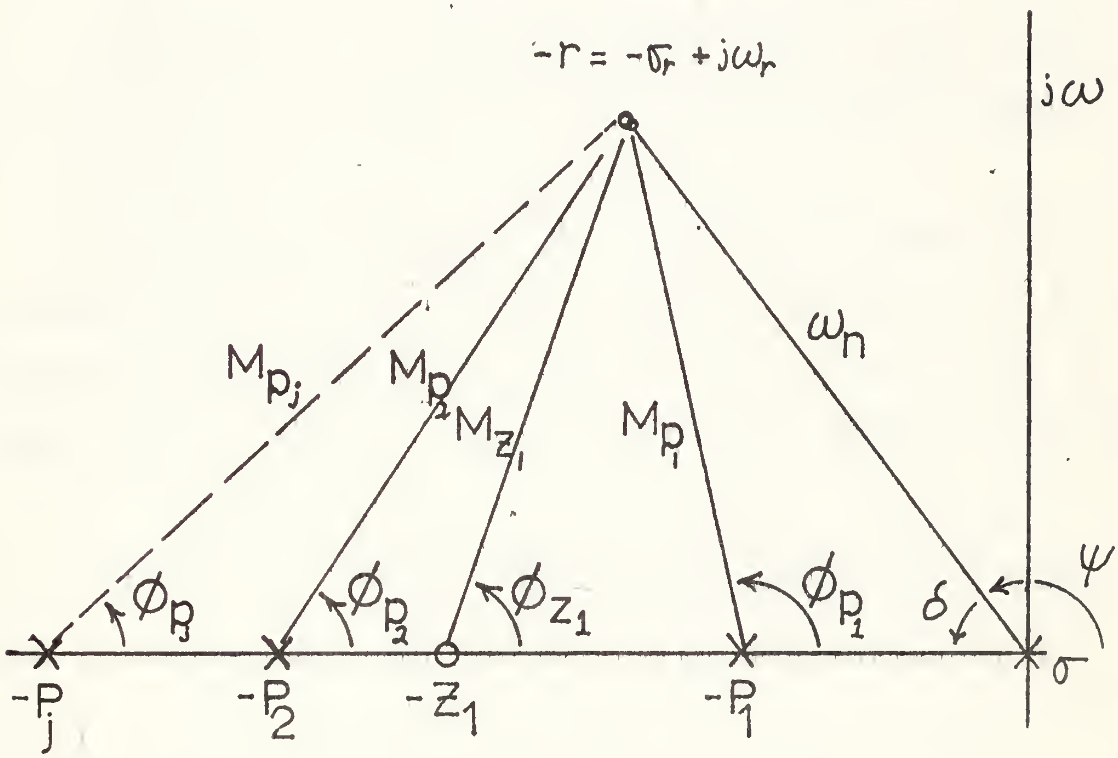
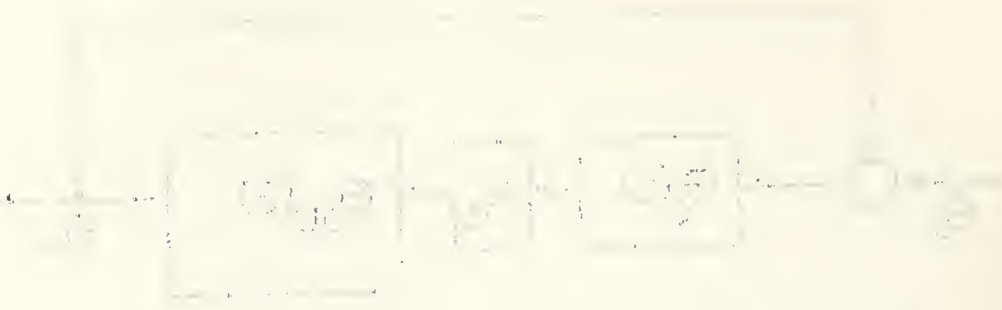


FIGURE 2-2 POLES AND ZEROS OF $G_{ou}(S)$ AS RELATED TO THE POINT $S = -r$



for applying compensation in this manner. The discussion of possible areas of the s plane in which a root may be located by this system does depend upon accurate computation of the uncompensated system roots. The question of "root re-location zones" will be considered separately. It will be assumed in this portion of the discussion that a unique solution for the pole and zero of the compensator network is in fact possible for the chosen point -r.

For various reasons a point on the s plane is now selected, say -r, and it is desired that (s + r) will be a factor of the characteristic equation of the compensated system. The primary reason for choosing -r if a hoped for transient response based upon knowledge of the response of a 2nd order system with corresponding complex roots. Within certain bounds the chosen root located at -r will indeed dominate the system transient response.

The compensated system referred to is to be derived from 2.1 by cascading a lead or lag filter of one or more identical sections. The simplified block diagram of the compensated system is then shown in Figure 2 and the open loop transfer function of the compensated system is

2.3 $$k_c G_{oc} = \frac{k_c\left[(S+Z_1)\cdots(S+Z_i)\right]\ (S+Z_c)^{\eta}}{S^n\left[(S+P_1)\cdots(S+P_j)\right]\ (S+P_c)^{\eta}}$$

Where n is the system type number and η is the number of identical sections of lead or lag network being employed.

The factor $\dfrac{(S+Z)^{\eta}}{(S+P)^{\eta}}$ may be considered as a lead or lag network, the subscript ℓ or χ being used to denote this fact. (The value of K_c depends upon the type of compensator).

The attack on the problem will be the usual one of root locus analysis. The "gain" or magnitude criterion will be applied first, and then the angle criterion will be applied. The result will be an equation involving both the magnitude and angle criterion applied at -r.

<u>Gain criterion applied at -r</u>:

Consider the symbols defined by Figure 2-2. Figure 2-2 presents some of the graphical parameters associated with the uncompensated system and the point -r. The uncompensated system, 2.1 has a gain coefficient k_u which has been specified. It is a part of the "plant". From the graphical terminology of root locus theory k_u can be evaluated at a root by forming the ratio of the product of open loop pole distances to the root divided by the open loop zero distances to the same root. In a similar manner, as suggested by Figure 2-2 a gain coefficient for the uncompensated system can be defined at the arbitrarily chosen point -r. This new gain coefficient will be denoted as G or G_{-r}. Where

2.4 $$G \equiv G_{-r} \equiv \frac{\prod_{K=1}^{j} M_{P_K}}{\prod_{K=1}^{i} M_{Z_K}}$$

G can be evaluated directly from the root locus plot of Equation 2.1.

After the compensator has been applied, the gain coefficient can again be evaluated at the point -r. This coefficient has been denoted k_c (See Equation 2.2). Where at -r :

2.5 $$k_c = \frac{\prod_{K=1}^{j} M_{P_K}}{\prod_{K=1}^{i} M_{Z_K}} \left(\frac{M_{P_\ell}}{M_{Z_\ell}}\right)^{\eta} = G_{-r}\left(\frac{M_{P_\ell}}{M_{Z_\ell}}\right)^{\eta}$$

The form for the error coefficient after compensation is, (From Equation 2.3);

2.6 $$K_c = \lim_{S \to 0} S^n k_c G_{oc}(S) = k_c \frac{\prod z_i}{\prod P_j} \left(\frac{z_\ell}{P_\ell}\right)^{\eta}$$

and substituting for k_c from Equation 2.5;

2.7 $$K_c = G_{-r} \frac{\prod z_i}{\prod P_j} \left(\frac{z_\ell}{P_\ell}\right)^{\eta} \left(\frac{M_{P_\ell}}{M_{z_\ell}}\right)^{\eta}$$

The design constraint is applied at this point in the form;

2.8 $$K_u = K_c$$

That is, the error coefficient is to be the same before and after compensation. Equation 2.8 states that the right hand sides of Equations 2.2 and 2.7 may be equated to form:

2.9 $$k_u \frac{\prod z_i}{\prod P_j} = G_{-r} \frac{\prod z_i}{\prod P_j} \left(\frac{z_\ell}{P_\ell} \frac{M_{P_\ell}}{M_{z_\ell}}\right)^{\eta}$$

And by cancellation and rearranging 2.9 becomes

2.10 $$\frac{z_\ell}{P_\ell} \frac{M_{P_\ell}}{M_{z_\ell}} = \left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}$$

Equation 2.10 is a key relation in the argument leading to an explicit solution for Z_L and P_L. At this point Z_L, P_L, M_{ZL}, M_{PL} and η are unknown, but the ratio $\frac{z_\ell M_{P_\ell}}{P_\ell M_{z_\ell}}$ is known in terms of k_u, the uncompensated system gain coefficient; G_{-r}, the uncompensated system gain evaluated at -r; and η, the number of identical phase lead

networks to be employed. This ratio could be denoted as $\alpha_\ell k_\ell$, where α_ℓ is the ratio $\frac{Z_\ell}{P_\ell}$ and k_ℓ is the ratio $\frac{M_{P\ell}}{M_{Z\ell}}$. At the present there is little to be gained by such a substitution other than brevity.

Before going on to consider the root locus angle criterion at -r it will be convenient to investigate the form that Equation 2.10 will take when a multiple section lag network is employed. Examination of Equations 2.5, 2.6, 2.7, 2.9, and 2.10 indicates that if the compensator to be employed is indeed a phase lag type, there will be no change in the form of the equation. That is, the derivation of the $\alpha_\ell k_\ell$ ratio in 2.10 does not depend on the relative magnitude of the compensator pole and zero. Therefore it can immediately be stated that;

2.11 $$\frac{Z_x}{P_x}\frac{M_{Px}}{M_{Z,x}} = \alpha_x k_x = \left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}$$

This result will be referred to later.

In considering phase angle relations for the compensator network at the point -r it will turn out that the relative magnitudes of the pole and zero co-ordinates must be taken into account. Therefore, Figure 2-3 for the phase lead network will be considered first. In Figure 2-3 $\omega_n = |-r|$ and the usual definition of damping ratio for a dominant root in this case would imply $\delta = \cos^{-1}\zeta$. A plane trigonometry solution based on triangles $O, -r, -P_\ell$ and $O, -r, -Z_\ell$ will be made for two unknown sides, P_ℓ, Z_ℓ. The six sides of the triangles will be designated $\omega_n, M_{P\ell}, P_\ell, \omega_n, M_{Z\ell}, Z_\ell$ respectively. The symbols chosen for the various angles are arbitrary and the diagram serves to define them except as noted below. δ and ψ are of course known immediately when the point -r has been chosen.

η is the multiplicity of identical compensator sections; and

infers that at the point $-P_L$ there are η poles and at the point $-Z_L$ there are η zeros.

ϕ is defined to be the angle that must be supplied by the η section filter. It is a known angle in that the phase angle locus of the uncompensated system that passes through -r can be determined. Call this phase angle locus the " ϕ_1 locus of $k_u G_u(S)$ ". ϕ_1 can be quickly evaluated at -r with the aid of a <u>spirule</u> at the same time that G_{-r} is evaluated. If -r is to be a point on the root locus of the compensated system, 2.3, then

2.12 $$\phi + \phi_1 = \pm m\pi$$

where m is an odd integer, (usually 1). Equation 2.12 can be considered to be the definition of ϕ_1 in terms of $\frac{\phi}{\eta}$. Therefore $\frac{\phi}{\eta}$ is a known quantity in the problem.

The embarrassing problem of the sign convention associated with Equation 2.12 will be sidestepped by stipulating that in this discussion $\frac{\phi}{\eta}$ will always be taken as a positive number defined in terms of the compensator as

2.13 $$|\phi_{Z_\ell}| - |\phi_{P_\ell}| = \left|\frac{\phi}{\eta}\right|$$

λ_ℓ is at present unknown. The solution for λ_ℓ will be an intermediate step in the solution for P_ℓ and Z_ℓ.

The law of sines from plane trigonometry may now be applied to the triangles of Figure 2a. For triangle $O, -r, -P_\ell$

2.14 $$\frac{M_{P_\ell}}{\sin S} = \frac{P_\ell}{\sin \lambda} = \frac{\omega_n}{\sin \phi_{P_\ell}}$$

and for triangle $O, -r, -Z_\ell$ the law of sines states that;

$$\frac{M_{Z_\ell}}{\sin \delta} = \frac{Z_\ell}{\sin\left(\lambda_\ell - \frac{\phi}{2}\right)} = \frac{\omega_n}{\sin \phi_{Z_\ell}} \tag{2.15}$$

Dividing the left hand equality of Equation 2.14 by the corresponding portion of Equation 2.15 produces a new relation in which a value previously determined appears.

$$\frac{M_{P_\ell}}{M_{Z_\ell}} = \frac{P_\ell}{Z_\ell} \times \frac{\sin\left(\lambda_\ell - \frac{\phi}{2}\right)}{\sin \lambda_\ell} \tag{2.16}$$

Equation 2.16 may be rearranged as

$$\frac{Z_\ell}{P_\ell}\,\frac{M_{P_\ell}}{M_{Z_\ell}} = \frac{\sin\left(\lambda_\ell - \frac{\phi}{2}\right)}{\sin \lambda_\ell} \tag{2.17}$$

Since Equation 2.10 evaluates the ratio $\frac{Z_\ell}{P_\ell}\,\frac{M_{P_\ell}}{M_{Z_\ell}}$ in terms of the specified uncompensated system parameters, Equation 2.17 therefore involves only one unknown. Therefore:

$$\frac{\sin\left(\lambda_\ell - \frac{\phi}{2}\right)}{\sin \lambda_\ell} = \left(\frac{k_u}{G_{\Omega}}\right)^{\frac{1}{2}} \tag{2.18}$$

By expanding and rearranging 2.18 in the following steps a more explicit solution for λ_ℓ may be made.

2.19 $$\frac{\sin\lambda_\ell \cos\frac{\phi}{\eta} - \cos\lambda_\ell \sin\frac{\phi}{\eta}}{\sin\lambda_\ell} = \left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}$$

2.20 $$\cos\frac{\phi}{\eta} - \frac{\sin\frac{\phi}{\eta}}{\tan\lambda_\ell} = \left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}$$

2.21 $$\tan\lambda_\ell = \frac{\sin\frac{\phi}{\eta}}{\cos\frac{\phi}{\eta} - \left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}}$$

If it is desired to solve for λ_ℓ numerically Equation 2.21 is helpful in that it is possible to observe the sign of numerator and denominator and thus avoid any confusion about the proper quadrant for λ_ℓ .

Having a solution for λ_ℓ it is now possible to solve for P_L using the right hand equality of equation .14

2.22 $$\frac{P_\ell}{\sin\lambda_\ell} = \frac{\omega_n}{\sin\phi_{P_\ell}} \quad \text{OR} \quad P_\ell = \frac{\omega_n \sin\lambda_\ell}{\sin\phi_{P_\ell}}$$

From Figure 2-3 it is seen that

2.23 $$\phi_{P_\ell} = \pi - \lambda_\ell - \delta$$

So that:

2.24 $$P_\ell = \frac{W_m \sin \lambda_\ell}{\sin(\lambda_\ell + \delta)}$$

2.25 $$P_\ell = \frac{W_m \sin \lambda_\ell}{\sin \lambda_\ell \cos \delta + \cos \lambda_\ell \sin \delta}$$

2.26 $$P_\ell = W_m \frac{\tan \lambda_\ell}{\tan \lambda_\ell \cos \delta + \sin \delta}$$

Replacing $\tan \lambda_\ell$ in 2.26 by its equivalent from Equation 2.21.

2.27 $$P_\ell = \frac{W_m \dfrac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2} - \left(\frac{k}{G}\right)^{\frac{1}{2}}}}{\dfrac{\sin \frac{\phi}{2} \cos \delta + \left[\cos \frac{\phi}{2} - \left(\frac{k}{G}\right)^{\frac{1}{2}}\right]}{\cos \frac{\phi}{2} - \left(\frac{k}{G}\right)^{\frac{1}{2}}}}$$

And 2.27 reduces to

2.28 $$P_\ell = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta}+\delta\right) - \left(\frac{k_u}{G_n}\right)^{\frac{1}{\eta}} \sin\delta}$$

Which is an explicit solution for P_ℓ in terms of the specified uncompensated system parameters and the chosen dominant root pair location.

The right hand equality of Equation 2.15 leads to a solution for Z_ℓ as follows:

2.29 $$\frac{Z_\ell}{\sin\left(\lambda_\ell - \frac{\phi}{\eta}\right)} = \frac{\omega_n}{\sin\phi_{Z_\ell}}$$

So that:

2.30 $$Z_\ell = \omega_n \frac{\sin\left(\lambda_\ell - \frac{\phi}{\eta}\right)}{\sin\phi_{Z_\ell}}$$

From Figure 2-3 it is noted that

2.31 $$\phi_{Z_\ell} = \pi - \left(\lambda_\ell - \frac{\phi}{\eta} + \delta\right)$$

and therefore

2.32 $$Z_\ell = \omega_n \frac{\sin\left(\lambda_\ell - \frac{\phi}{\eta}\right)}{\sin\left[\left(\lambda_\ell - \frac{\phi}{\eta}\right) + \delta\right]}$$

Expanding 2.32 it is seen that:

$$2.33 \quad Z_\ell = \omega_n \frac{\sin\left(\lambda_\ell - \frac{\phi}{\eta}\right)}{\sin\left(\lambda_\ell - \frac{\phi}{\eta}\right)\cos\delta + \cos\left(\lambda_\ell - \frac{\phi}{\eta}\right)\sin\delta}$$

Dividing numerator and denominator of 2.33 by $\cos\left(\lambda_\ell - \frac{\phi}{\eta}\right)$;

$$2.34 \quad Z_\ell = \omega_n \frac{\tan\left(\lambda_\ell - \frac{\phi}{\eta}\right)}{\tan\left(\lambda_\ell - \frac{\phi}{\eta}\right)\cos\delta + \sin\delta}$$

But, $\tan\left(\lambda_\ell - \frac{\phi}{\eta}\right)$ may be solved for as an intermediate step:

$$2.35 \quad \tan\left(\lambda_\ell - \frac{\phi}{\eta}\right) = \frac{\tan\lambda_\ell - \tan\frac{\phi}{\eta}}{1 + \tan\lambda_\ell \tan\frac{\phi}{\eta}}$$

In Equation 2.35 $\tan\lambda_\ell$ may be replaced by the right hand side of Equation 2.21.

The result of this replacement is:

$$2.36 \quad \tan\left(\lambda_\ell - \frac{\phi}{\eta}\right) = \frac{\sin\frac{\phi}{\eta}\cos\frac{\phi}{\eta} - \left[\cos\frac{\phi}{\eta} - \left(\frac{k}{G}\right)^{\frac{1}{\eta}}\right]\sin\frac{\phi}{\eta}}{\cos^2\frac{\phi}{\eta} - \left(\frac{k}{G}\right)^{\frac{1}{\eta}}\cos\frac{\phi}{\eta} + \sin^2\frac{\phi}{\eta}}$$

And 2.36 can be simplified in the two following steps.

2.37 $$\tan\left(\lambda_l - \frac{\phi}{2}\right) = \frac{\sin\frac{\phi}{2}\cos\frac{\phi}{2} - \left[\cos\frac{\phi}{2} - \left(\frac{k}{G}\right)^{\frac{1}{2}}\right]\sin\frac{\phi}{2}}{\cos^2\frac{\phi}{2} - \left(\frac{k}{G}\right)^{\frac{1}{2}}\cos\frac{\phi}{2} + \sin^2\frac{\phi}{2}}$$

$$= \frac{\left(\frac{k}{G}\right)^{\frac{1}{2}}\sin\frac{\phi}{2}}{1 - \left(\frac{k}{G}\right)^{\frac{1}{2}}\cos\frac{\phi}{2}}$$

2.38 $$\tan\left(\lambda_l - \frac{\phi}{2}\right) = \frac{\sin\frac{\phi}{2}}{\left(\frac{G_n}{k_n}\right)^{\frac{1}{2}} - \cos\frac{\phi}{2}}$$

With the results of Equation 2.38, Equation 2.34 can be reduced as follows:

2.39 $$Z_l = \frac{\omega_n \dfrac{\sin\frac{\phi}{2}}{\left(\frac{G}{k}\right)^{\frac{1}{2}} - \cos\frac{\phi}{2}}}{\dfrac{\sin\frac{\phi}{2}}{\left(\frac{G}{k}\right)^{\frac{1}{2}} - \cos\frac{\phi}{2}} + \sin\delta}$$

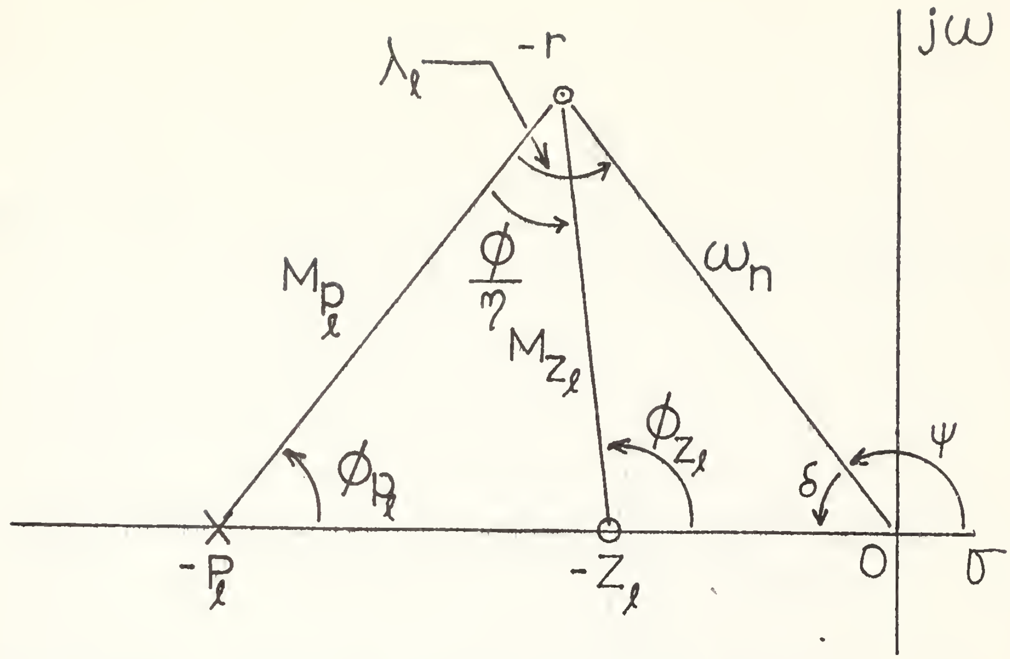


FIGURE 2-3 PHASE LEAD NETWORK

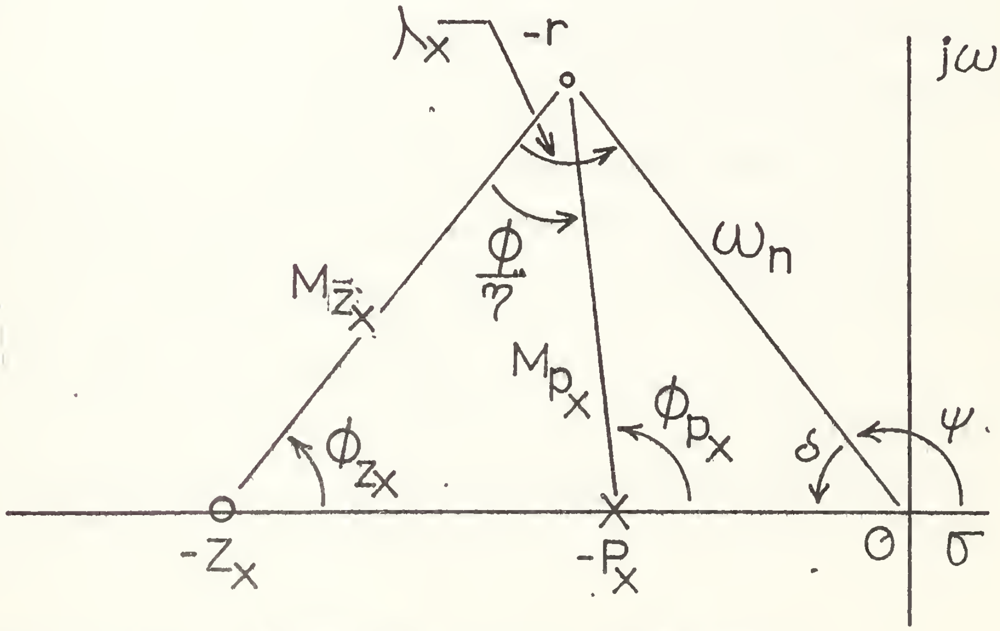


FIGURE 2-4 PHASE LAG NETWORK

2.40 $$Z_\ell = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\frac{\phi}{\eta}\cos\delta + \left(\frac{G}{k}\right)^{\frac{1}{\eta}}\sin\delta - \cos\frac{\phi}{\eta}\sin\delta}$$

2.41 $$Z_\ell = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta}-\delta\right) + \left(\frac{G-r}{k_u}\right)^{\frac{1}{\eta}}\sin\delta}$$

Equation 2.41 is a companion solution to that of Equation 2.28. The form of 2.28 and 2.41 is very similar. Note the interchange of signs and the reciprocal of the $\frac{k}{G}$ ratio.

THE PHASE LAG NETWORK

Equations 2.28 and 2.41 represent explcit solutions for the poles and zeros of an η section phase lead network. A very similar solution for the poles and zeros of ansection phase lag network should be expected from the symmetry of the problem. Where possible the form of the phase lead equations will be appealed to in finding the phase lag solution. As stated before, the question of the existence of real positive solutions for a given choice of the point -r will be deferred to the section on "root relocation zones".

Referring to Figure 2-4 the law of sines applied respectively to triangles $O_j - r_j - Z_x$ and $O_j - r_j - P_x$ produces the following two relations;

2.42 $$\frac{M_{z_x}}{\sin \delta} = \frac{Z_x}{\sin \lambda_x} = \frac{C\omega_n}{\sin \phi_{z_x}}$$

2.43 $$\frac{M_{P_x}}{\sin \delta} = \frac{P_x}{\sin\left(\lambda_x - \frac{\phi}{\eta}\right)} \frac{C\omega_n}{\sin \phi_{P_x}}$$

Note that λ_x is defined differently in Figure 2-3 than λ_x was defined in Figure 2-2.

As before the angle ϕ will be defined as the angle that must be provided by the compensator network so that -r will be a point on a root locus segment of the compensated system. The direction of measurement of $\frac{\phi}{\eta}$ on diagram 2-3 is the conventional positive direction. The geometry of the figure defines $\frac{\phi}{\eta}$ in terms of ϕ_{z_x} and ϕ_{P_x} as;

2.44 $$\left|\phi_{P_x}\right| - \left|\phi_{z_x}\right| = \left|\frac{\phi}{\eta}\right|$$

Again the absolute value symbols are used only to avoid the question of the sign convention for pole and zero angles. The following argument assumes $\frac{\phi}{\eta}$ to be a positive number. (Equation 2.12 will assign a plus or minus sign to ϕ depending on the convention used.) In any event, $\frac{\phi}{\eta}$ will be less in absolute value than ψ .

Dividing the left hand equality of Equation 2.42 by the corresponding portion of Equation 2.43 it is found that;

2.45 $$\frac{M_{z_x}}{M_{p_x}} = \frac{Z_x}{P_x} \frac{\sin(\lambda_x - \frac{\phi}{2})}{\sin \lambda_x}$$

Equation 2.45 can be rearranged as

2.46 $$\frac{P_x}{Z_x} \frac{M_{z_x}}{M_{p_x}} = \frac{\sin(\lambda_x - \frac{\phi}{2})}{\sin \lambda_x}$$

By reference to Equation 2.11 it is possible to write an equation in which λ_x is the single unknown.

2.47 $$\frac{\sin(\lambda_x - \frac{\phi}{2})}{\sin \lambda_x} = \left(\frac{G_{-n}}{k_u}\right)^{\frac{1}{\eta}}$$

Equation 2.47 compares in form to Equation 2.18 and this fact may be used to write immediately the solution for λ_x using the form of Equation 2.21.

2.48 $$\tan \lambda_x = \frac{\sin \frac{\phi}{\eta}}{\cos \frac{\phi}{\eta} - \left(\frac{G_{-n}}{k_u}\right)^{\frac{1}{\eta}}}$$

Using the right hand equality of Equation 2.42 a solution for Z_x can be written.

2.49 $$Z_x = \omega_n \frac{\sin \lambda_x}{\sin \phi_{z_x}}$$

And from Figure 2b it can be seen that;

2.50 $$\phi_{Z_x} = \pi - \lambda_x - \delta$$

Again the form of a preceding solution may be used.

Examination of the form of Equation 2.21, 2.22, 2.23 shows that Equation 2.28 has the form of the desired solution.

The replacements to be made are:

P_ℓ replaced by Z_x (Compare 2.22 and 2.49
" 2.23 and 2.50)

$\left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}$ replaced by $\left(\frac{G_{-n}}{k_u}\right)^{\frac{1}{\eta}}$ (Compare 2.21 and 2.48)

So that the solution for Z_x is:

2.51 $$Z_x = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta}+\delta\right) - \left(\frac{G_{-n}}{k_u}\right)^{\frac{1}{\eta}} \sin\delta}$$

Finally, a solution for P_x can be obtained by use of the right hand equality of Equation 2.43

2.52 $$P_x = \omega_n \frac{\sin\left(\lambda_x - \frac{\phi}{\eta}\right)}{\sin\phi_{P_x}}$$

And from Figure 2-4 it can be seen that

2.53 $$\phi_{P_x} = \pi - \left(\lambda_x - \frac{\phi}{\eta} + \delta\right)$$

Comparing the form of Equations 2.52 and 2.30

2.53 and 2.31

2.48 and 2.21 it is

possible to use the form of Equation 2.41 for the solution with the replacements:

Z_λ replaced by P_x

$\left(\frac{G}{k}\right)^{\frac{1}{\eta}}$ replaced by $\left(\frac{k}{G}\right)^{\frac{1}{\eta}}$.

So that the solution for P_x becomes:

2.54 $$P_x = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta} - \delta\right) + \left(\frac{k_4}{G_n}\right)^{\frac{1}{\eta}} \sin\delta}$$

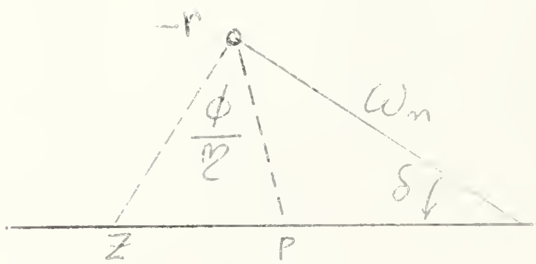
The results of the foregoing argument may be summarized by listing the four "compensator equations". See Figure 2-5 for a typical example of computations based on the compensator equations.

2.28 $$P_l = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta}+\delta\right)-\left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}\sin\delta}$$

2.41 $$Z_l = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta}-\delta\right)+\left(\frac{G_{-r}}{k_u}\right)^{\frac{1}{\eta}}\sin\delta}$$

2.54 $$P_x = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta}-\delta\right)+\left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}\sin\delta}$$

2.51 $$Z_x = \omega_n \frac{\sin\frac{\phi}{\eta}}{\sin\left(\frac{\phi}{\eta}+\delta\right)-\left(\frac{G_{-r}}{k_u}\right)^{\frac{1}{\eta}}\sin\delta}$$



EXAMPLE OF RESULTS OF APPLICATION OF COMPENSATOR EQUATIONS AT A POINT $-r= -3+j4$ LOCATED IN A SINGLE LEAD NETWORK ZONE. SOLUTIONS FOR P_ℓ AND Z_ℓ

$\eta = 1,2,3,4,5,6.$

η	1	2	3	4	5	6
P_ℓ	6.02	4.22	3.78	3.57	3.45	3.37
Z_ℓ	1.30	2.07	2.36	2.52	2.61	2.68
α	.217	.492	.628	.705	.757	.795
	α^2 .242					
		α^3 .246				
			α^4 .247			
				α^5 .248		
					α^6 .250	

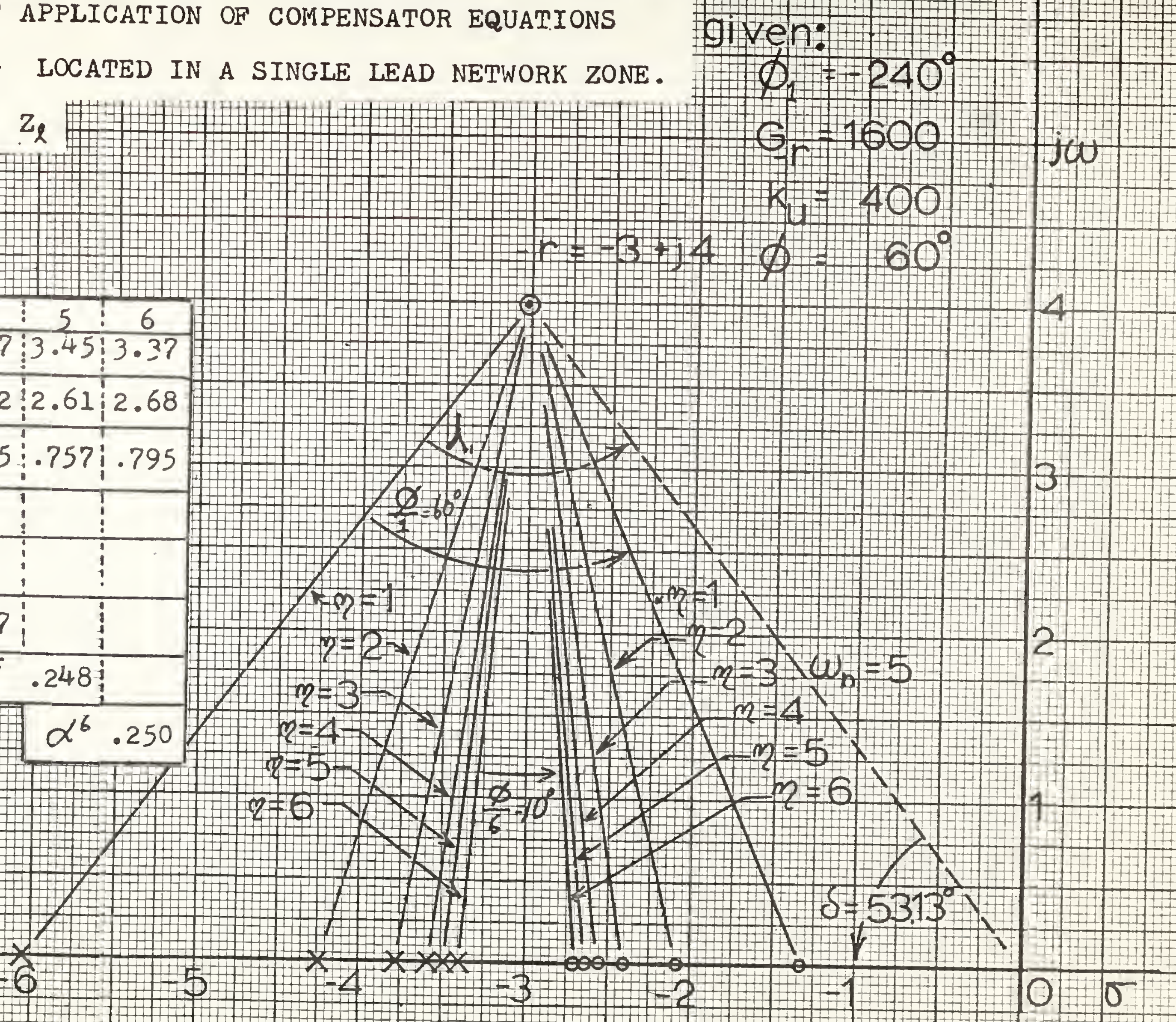


FIGURE 2-5

3. ATTENUATION FACTOR LIMIT FOR MULTIPLE SECTION COMPENSATORS

The attenuation for an η identical section compensator network is:

$$\alpha^{\eta} = \left(\frac{Z_c}{P_c}\right)^{\eta} \tag{3.1}$$

where $0 < \alpha^{\eta} < 1$ for a lead network and $1 < \alpha^{\eta}$ for a lag network.

From the compensator equations it is possible to compute α for a lead or lag network as follows:

Divide Equation 2.28 by Equation 2.41 so that, for the lead network;

$$\alpha_{l} = \frac{Z_l}{P_l} = \frac{\sin\left(\frac{\phi}{\eta} + \delta\right) - \left(\frac{k}{G}\right)^{\frac{1}{\eta}} \sin\delta}{\sin\left(\frac{\phi}{\eta} - \delta\right) + \left(\frac{G}{k}\right)^{\frac{1}{\eta}} \sin\delta} \tag{3.2}$$

Similarly for the lag network divide Equation 2.51 by Equation 2.54 to produce;

$$\alpha_{r} = \frac{Z_r}{P_r} = \frac{\sin\left(\frac{\phi}{\eta} - \delta\right) + \left(\frac{k}{G}\right)^{\frac{1}{\eta}} \sin\delta}{\sin\left(\frac{\phi}{\eta} + \delta\right) - \left(\frac{G}{k}\right)^{\frac{1}{\eta}} \sin\delta} \tag{3.3}$$

The example in the preceding section (Figure 2-5) suggests that there may be an advantage to using several identical sections to compensate to a given point rather than a single section (when -r is in a single section zone). One method for studying the attenuation ratio, α^{η}, is to consider Equations 3.2 and 3.3 as η increases. Taking Equation 3.2 first this process may be stated as:

$$3.4 \quad \lim_{\eta\to\infty} \alpha_\ell = \lim_{\eta\to\infty} \frac{\sin\left(\frac{\phi}{\eta}+\delta\right) - \left(\frac{k}{G}\right)^{\frac{1}{\eta}} \sin\delta}{\sin\left(\frac{\phi}{\eta}-\delta\right) + \left(\frac{G}{k}\right)^{\frac{1}{\eta}} \sin\delta}$$

In the limit $\left(\frac{\phi}{\eta}+\delta\right) \to \delta$ as $\eta \to \infty$

and $\left(\frac{\phi}{\eta}-\delta\right) \to -\delta$ as $\eta \to \infty$

These two substitutions will be made. However, $\left(\frac{k}{G}\right)^{\frac{1}{\eta}}$ will not be replaced by unity as would be normal in such a limiting process. Such a procedure can be justified by a formal investigation of the convergence of the limiting process. In this case experience and intuition indicates that the contribution of $\frac{\phi}{\eta}$ becomes small relative to δ very rapidly but since we are to raise the resulting ratio to the η^{th} power the $\left(\frac{k}{G}\right)^{\frac{1}{\eta}}$ term retains its significance. With these ideas in mind, Equation 3.4 can be written as;

$$3.5 \quad \lim_{\eta\to\infty} \alpha_\ell = \frac{\sin\delta - \left(\frac{k}{G}\right)^{\frac{1}{\eta}} \sin\delta}{\sin(-\delta) + \left(\frac{G}{k}\right)^{\frac{1}{\eta}} \sin\delta}$$

In Equation 3.5 the $\sin\delta$ term can be factored out and canceled to produce;

$$3.6 \quad \lim_{\eta\to\infty} \alpha_\ell = \frac{1 - \left(\frac{k}{G}\right)^{\frac{1}{\eta}}}{-1 + \left(\frac{G}{k}\right)^{\frac{1}{\eta}}}$$

Multiplying numerator and denominator of 3.6 by $\left(\frac{k}{G}\right)^{\frac{1}{\eta}}$

3.7 $$\lim_{\eta \to \infty} \alpha_l = \left(\frac{k}{G}\right)^{\frac{1}{\eta}} \frac{\left(1 - \left(\frac{k}{G}\right)^{\frac{1}{\eta}}\right)}{\left(1 - \left(\frac{k}{G}\right)^{\frac{1}{\eta}}\right)}$$

In Equation 3.7 the factor $\left(1 - \left(\frac{k}{G}\right)^{\frac{1}{\eta}}\right)$ can be cancelled and the result is:

3.8 $$\lim_{\eta \to \infty} \alpha_l = \left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}$$

Now if it is considered that the zero/pole ratio represented by 3.8 is a finite number equal to $\left(\frac{k}{G}\right)^{\frac{1}{\eta}}$ then the attenuation ratio for an η section filter, where η is an arbitrarily large but finite number is then:

3.9 $$\alpha_l^{\eta} = \left(\frac{k_u}{G_{-n}}\right)$$

(η large)

("large" is not defined. η = 4 might be considered "large enough" in most cases.)

As indicated in Figure 2-5 the attenuation ratio for a multiple section lead network would be improved by increasing the number of filter sections used to achieve a given complex pole location. The theoretical limit on the attenuation ratio with increasing η is $\frac{k_u}{G_{-n}}$.

It appears that the limit is approached rapidly, perhaps 2 or 3 sections coming close enough to the limit to preclude cascading additional units.

The corresponding result for the lag network, using Equation 3.13 is;

3.10 $$\lim_{\eta \to \infty} \alpha_x = \frac{-1 + \left(\frac{k}{G}\right)^{\frac{1}{\eta}}}{1 - \left(\frac{G}{k}\right)^{\frac{1}{\eta}}} = \left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}$$

From which it follows;

3.11 $$\alpha_x^{\eta} = \frac{k_u}{G_{-n}}$$

(η large)

A design procedure based on this concept would be to calculate a pole zero configuration based on the smallest value of η that will satisfy the problem conditions. The zero/pole ratio of the computed compensator may then be compared with the ratio $\frac{k_u}{G_{-n}}$. If the comparison indicates that an improvement in the attenuation ratio is possible and practical, then a higher value of η can be used to calculate a new pole zero configuration for the compensator.

4. ROOT RELOCATION ZONES: GENERAL DISCUSSION OF DEFINING FUNCTIONS FOR THE LIMIT LOCI; SINGLE SECTION ZONES

In the preceding derivation of the compensator equations it was assumed that a solution for a compensator pole zero configuration was possible for the selected location of a dominant root, -r. Ross and Warren found that the entire s plane could be divided into "lead and lag" zones with the $\pm\pi$ root loci of the uncompensated system serving as boundaries. However, it was found that the number of sections of compensator network required for a solution was theoretically (impractically) high except for well defined "single lead or lag zones". The method derived by Ross and Warren to define the single Section zones will be developed in a slightly different manner here to bring out the relation of the re-location zone limit lines to the specified uncompensated system transfer function. It will be indicated in Section 6 that in theory this method can be applied to define the limit lines for double, triple, etc., zones. However, for $\eta > 1$ the procedure requires a root locus solution for each point of the limit.

The argument which follows is stated emplicitly in the thesis by Ross and Warren [8] but a proof based on the theory of equations was used. The general form of the derivation and the notation used here was suggested by Dr. R. C. H. Wheeler in his lectures on advanced linear servo theory. The extension to zones in which compensator attenuation is held above a specified minimum was outlined by Dr. G. J. Thaler. Certain additional points were suggested by W. Evans (See Evans, Control-System Dynamics, page 160) [1].

The open loop transfer function of an uncompensated system

$$k_u G_{ou}(S) = \frac{k_{ou}\,(S+Z_1)\cdots(S+Z_i)}{(S+P_1)\cdots(S+P_j)}$$

4.1 $$\text{where } K_u = k_{ou}\,\frac{Z_1\cdots Z_i}{P_1\cdots P_j}\quad ;\ k_u = k_{ou}$$

The form is restricted to Type 0 or higher, that is,

P_1, P_2 ... may all be greater than zero, or one or more of the P_j may be zero, but the exponent of s in the numerator cannot be negative.

The closed loop transfer function of the uncompensated system can then be written as

4.2 $$k_u G_{cu}(S) = \frac{k_{cu}[(S+Z_1)\cdots(S+Z_i)]}{[(S+P_1)\cdots(S+P_j)] + k_{ou}[(S+Z_1)\cdots(S+Z_i)]}$$

or

4.3 $$k_u G_{cu}(S) = \frac{k_{ou}[(S+Z_1)\cdots(S+Z_i)]}{[(S+r_1)\cdots(S+r_j)]}$$

Where the r_j are the roots of the system characteristic equation. This form, written in terms of the roots of the system characteristic equation implies that the coefficient of s^j is unity, or in other words $j \geq i + 1$. That is, the order of the denominator must be at least one greater than the numerator.

If the system described by 4.1 is compensated by cascading a single section lead network, the following open loop transfer function can be written.

4.4 $$k_c G_{oc}(S) = \frac{k_{ou}[(S+Z_1)\cdots(S+Z_i)]}{[(S+P_1)\cdots(S+P_j)]} \frac{(S+Z_\ell)}{(S+P_\ell)}$$

Where

4.5 $$K_c = k_c \frac{Z_1 \cdots Z_i}{P_i \cdots P_j} \frac{Z_\ell}{P_\ell}$$

As in the derivation of the "compensator equations" the restriction that the error coefficient be invarient must be applied. (As stated before, some other restriction could be used, but a constant K is the most practical.) If the system K is to remain unchanged, the system type number can not change. This means that $|Z_\ell| > 0$, from which it follows that $|P_\ell| > 0$. $|Z_\ell|$ May approach 0 only in a limiting process.

If

4.6 $$K_u = K_c$$

Where K_u and K_c are defined in 4.1 and 4.5.

Then;

4.7 $$k_u \frac{Z_1 \cdots Z_i}{P_1 \cdots P_j} = k_c \frac{Z_1 \cdots Z_i \times Z_\ell}{P_1 \cdots P_j \times P_\ell}$$

Or;

4.8 $$k_c = k_u \frac{P_\ell}{Z_\ell} \quad \text{AND} \quad P_\ell > Z_\ell$$

And it can be noted at this point that for a single section lag network the corresponding relation is:

$$4.9 \quad k_c = k_u \frac{P_x}{Z_x} \quad \text{WHERE} \quad P_x < Z_x$$

Equations 4.1, 4.4 and 4.8 can be combined as;

$$4.10 \quad k_c G_{oc}(S) = \left(\frac{P_l}{Z_l} \frac{S+Z_l}{S+P_l} \right) k_u G_{ou}(S)$$

The limits of a single section lead network can now be considered as a situation in which P_L recedes to infinity and Z_L takes on particular finite values. Or, for an finite Z_L the Maximum phase angle compensation that can be obtained is that of a network with an infinitely large value of P_L. It is natural then to investigate the locus of "limit points" for the system in which P_L is held at infinity and Z_L is allowed to assume all values between $+\infty$ and 0. (The zero "travels" from $-\infty$ to $-\epsilon < 0$) It must be noted that the crucial point of the argument depends upon 4.6, the invariance of the system error coefficient during the compensation process. Using the general technique of using the root locus method to observe the effect of varying one parameter in an equation; it is now possible to produce a transfer function in which Z_L is the variable and P_L is held at infinity. It should then be possible to interpret the root locus of this system as a boundary of the "single lead zone" of the s plane.

Equation 4.10 can therefore be considered as an expression which, in the limiting case of $P_L \rightarrow \infty$; defines a boundary of application for the single section lead network.

This limiting case will be denoted as $\lim\limits_{P_l \to \infty} k_c G_{oc}(S)$

and a similar notation will be used for the closed loop transfer function of the compensated system. This function will be denoted $k_c G_{cc}(S)$ where the subscripts mean "closed loop" and "compensated". The corresponding limiting case will be denoted $\lim\limits_{P_l \to \infty} k_c G_{cc}(S)$

In 4.10 the parameters P_L and Z_L do not appear in $k_u G_u(S)$ so it is safe to examine the system in which P_L becomes infinite. While Z_L remains finite in the following manner:

4.11 $$\lim_{P_l \to \infty} k_c G_{oc}(S) = \lim_{P_l \to \infty} \frac{1}{Z_l}\left(\frac{S + Z_l}{\frac{S}{P_l} + 1}\right) k_u G_u(S)$$

or

4.12 $$\lim_{P_l \to \infty} k_c G_{oc}(S) = \frac{S + Z_l}{Z_l} k_u \frac{(S+Z_1)\cdots(S+Z_j)}{(S+P_1)\cdots(S+P_j)}$$

From the limiting case presented by 4.12 we should now be able to determine the roots of $k_c G_{cc}(S)$ provided by a "maximum angle" lead network. Rather than manipulating 4.12 the relation being sought is more easily seen by writing out one form of $\lim\limits_{P_l \to \infty} k_c G_{cc}(S)$

based upon 4.12.

4.13 $$\lim_{P_l \to \infty} k_c G_{cc}(S) = \frac{\left(\frac{k_u}{Z_l}\right)\left[(S+Z_1)\cdots(S+Z_j)\right](S+Z_l)}{\left[(S+P_1)\cdots(S+P_j)\right] + \left(\frac{k_u}{Z_l}\right)\left[(S+Z_1)\cdots(S+Z_j)\right](S+Z_l)}$$

The roots of the characteristic equation are the roots of the denominator of $\lim k_c G_c(s)$ and we could proceed with 4.12 to define a root locus to find these roots, however, writing out the denominator of 4.13 it is seen that the equation to be solved can be formed as follows:

4.14 $$\left[(s+P_1)\cdots(s+P_j)\right] + \frac{k_u}{Z_\ell}\left[(s+Z_1)\cdots(s+Z_i)\right](s+Z_\ell) = 0$$

Equation 4.14 can be expanded to:

4.15 $$\left[(s+P_1)\cdots(s+P_j)\right] + \frac{k_u}{Z_\ell}\left[(s+Z_1)\cdots(s+Z_i)\right]s + \frac{k_u}{Z_\ell}\left[(s+Z_1)\cdots(s+Z_i)\right]Z_\ell = 0$$

Evans (Control System Dynamics, page 129ff) [1] outlines the general procedure or re-grouping factoring such and equation in an arbitrary fashion that allows consideration of any one of the parameters as the system variable. A convenient re-grouping of 4.15 is:

4.16 $$\left[(s+P_1)\cdots(s+P_j) + k_u(s+Z_1)\cdots(s+Z_i)\right] + \frac{k_u}{Z_\ell}\, s\left[(s+Z_1)\cdots(s+Z_i)\right] = 0$$

By reference to 4.3 it is seen that the term inside the brackets is precisely the characteristic equation of the uncompensated system's closed loop function. Therefore:

4.17 $$\left[(s+\Pi_1)\cdots(s+\Pi_j)\right] + \frac{k_u}{Z_\ell}\, s\left[(s+Z_1)\cdots(s+Z_i)\right] = 0$$

Equation 4.17 may now be put into the standard form for determination of a root locus for which $\frac{1}{Z_\ell}$ (or more precisely, $\frac{k_u}{Z_\ell}$) is the

system variable:

$$4.18 \qquad \left(\frac{k_{ou}}{Z_l}\right) \frac{S\left[(S+Z_1)\cdots(S+Z_1)\right]}{\left[(S+\Pi_1)\cdots(S+\Pi_j)\right]} = -1$$

Some observations may be made upon the general form of 4.18. First, it must be kept in mind that 4.18 describes the root locus defined from $k_c G_{oc}(S)$ corresponding to the roots of

$$\lim_{P_l \to \infty} k_c G_{oc}(S)$$

Second, the product in the numerator brackets represents the zeros of the original uncompensated system.

Third, the product in the denominator brackets represents the roots of the original uncompensated system.

Fourth, the reciprocal of Z_L appears as the variable and accounts for the relative "motion" of the limit for various finite values of Z_L.

From these observations a symbolic notation for the single section lead network limit line defined by a root locus associated with $\lim_{P_l \to \infty} k_c G_{oc}(S)$ is:

$$4.19 \qquad \left(\frac{k_u}{Z_l}\right) \frac{S\left[\text{ZEROS OF THE UNCOMPENSATED SYSTEM}\right]}{\left[\text{ROOTS OF THE UNCOMPENSATED SYSTEM}\right]} = -1$$

The root locus defined by 4.19 may then form the basis for a set of rules for determining the limit of application of the lead compensator equations when $P_L \to \infty$.

One such rule might be:

"To form the $P_L \to \infty$ limit line for a single section lead network,

construct a root locus based on the following singular points:

a.) $Z_\alpha = 0, z_1, \ldots, z_i$

b.) $P_\beta = \pi_1, \ldots, \pi_j$

the variable parameter being $\left(\frac{k_u}{Z_l}\right)$ "

It must be observed that this is exactly the list of Rules proposed by Ross and Warren, which are quoted for comparison. (R&W page 72) [8]

" a.) Delete all of the original poles.

b.) Retain all of the original zeros.

c.) Place a Zero at the origin, retaining any already there.

d.) Place poles at all of the original root positions.

e.) Draw a pseudo root locus using these new singular points." .

<u>Single Section Phase Lag Network</u>:

A similar limiting condition exists for the single section phase lag network. As was pointed out above one obvious limit on the lead and lag network zones is the original $\pm m\pi$ root locus of the uncompensated system. For example, to one "side" of the $\pm\pi$ locus the phase angle loci representing total system phase andle is greater in magnitude than π and represent a region in which a lead network is the obvious, and practical, type of compensation. Just "across" the locus from the lead area the magnitude of the uncompensated system phase andle is less than π and an infinitesimal distance from the $\pm\pi$ locus in this direction it is observed that a lag network of infitesimal pole-zero spacing is satisfactory to bring the root locus through a desired point. Again, as the point at which it is desired to establish a root moves away from the $\pm\pi$ locus in the direction of decreasing phase angle, a limiting position for the single

lag section, under the restriction of constant error coefficient, is approached. This limiting locus can be established for the single section lag network by the following argument.

It was noted above in passing (Equation 4.9) that the compensated system gain coefficient, K_c, is equal to the uncompensated system gain coefficient, K_{ou}, multiplied by the ratio of the lag network pole to the lag network zero, or;

4.9 $$k_c = k_{ou} \frac{P_x}{Z_x}$$

So the compensated system open loop transfer function can be expressed as

4.20 $$k_c G_{oc}(S) = \frac{P_x}{Z_x} \left(\frac{S + Z_x}{S + P_x} \right) k_u G_{ou}(S)$$

Now in this case the limit of a phase lag network is approached is Z_x approaches infinity for any particular finite value of P_x. As before two limiting case functions will be defined and the corresponding root locus will be examined, using P_x as the variable parameter:

4.21 $$\lim_{Z_x \to \infty} k_c G_{oc}(S) = \lim_{Z_x \to \infty} \frac{P_x \left(\frac{S}{Z_x} + 1 \right)}{(S + P_x)} k_u G_{ou}(S) =$$

$$\frac{P_x}{S + P_x} k_u G_{ou}(S) = \frac{P_x k_{ou} [(S + Z_1) \cdots (S + Z_i)]}{[(S + P_x)(S + P_1) \cdots (S + P_j)]}$$

And as before the limiting closed loop function is;

$$4.22 \quad \lim_{Z_x \to \infty} k_c G_{cc}(s) = \frac{P_x k_{ou}(s+Z_1)\cdots(s+Z_1)}{[(s+P_x)(s+P_1)\cdots(s+P_j)] + P_x k_{ou}[(s+Z_1)\cdots(s+Z_1)]}$$

The denominator of 4.22 serves as the basis for the root locus. (4.21 could have been used with refering to 4.22 as an intermediate step, but this way seems a bit clearer.) The characteristic equation of 4.22 is then;

$$4.23 \quad (s+P_x)[(s+P_1)\cdots(s+P_j)] + P_x k_{ou}[(s+Z_1)\cdots(s+Z_1)] = 0$$

Which may be re-grouped as;

$$4.24 \quad [s(s+P_1)\cdots(s+P_j)] + \{P_x[(s+P_1)\cdots(s+P_j) + k_{ou}(s+Z_1)\cdots(s+Z_1)]\} = 0$$

By inspection of 4.2 and 4.3 we see that 4.24 can be written as;

$$4.25 \quad [s(s+P_1)\cdots(s+P_j)] + P_x[(s+r_1)\cdots(s+r_j)] = 0$$

Where as before the r_j are the roots of the characteristic equation of the uncompensated system.

In the same manner that 4.18 was written, 4.25 may now be put in the "standard root locus form."

4.26 $$\frac{(P_x)\left[(S+n_1)\cdots(S+n_j)\right]}{S\left[(S+P_1)\cdots(S+P_j)\right]}\cdot e^{\pm j m\pi} = -1$$

And 4.26 may serve as the basis for a symbolic rule similar to 4.19.

4.27 $$\frac{(P_x)\left[\text{ROOTS OF THE UNCOMPENSATED SYSTEM}\right]}{S\left[\text{POLES OF THE UNCOMPENSATED SYSTEM}\right]} = -1$$

As before certain observations can be made on this form, notably, P_x is the variable and not the reciprocal as was the case for the variable in the lead network case. This fact should be kept in mind when analyzing the relative effect of varying parameters in the lag network. k_{ou} does not appear explicitly in the "variable term". Intuitively this is related to the fact that a lag network generally does not require adjustment of the system gain.

It should be born in mind that 4.26 merely describes a particular root locus associated with the compensated system when Z_x has been held at infinity. The compensated system in that case being described by the functions;

$$\lim_{Z_x \to \infty} k_c G_{oc}(S) \quad \text{AND} \quad \lim_{Z_x \to \infty} k_c G_{cc}(S)$$

A verbal rule based on 4.27 might take the form: "to form the $Z_x \to \infty$ limit line for a single section phase lag network, construct a root locus based on the following singular points.

a.) $Z_\alpha = Z_1, \ldots, Z_j$

b.) $P_\beta = 0, P_1, \ldots, P_j$

the variable parameter being (P_x)."

The rules set down by Ross and Warren are quoted for comparison.
(R&W page 72) [8]

" a.) Delete all of the original zeros.

b.) Retain all of the original poles.

c.) Place a pole at the origin, retaining any already there.

d.) Place zeros at all of the original root positions.

e.) Draw a pseudo root locus using these new singular points."

The rules are identical for all practical purposes. It might be better to think of the limit lines as a limiting condition of the compensated system root locus, rather than a "pseudo root locus."

A More Desirable Form For the Lag Network Limit Locus Equation:

It can be easily shown that the root locus plot of a ratio of complex polynomials such as Equation 4.26 is unchanged if the ratio is inverted. A quick check can be made by observing the root locus plot of a few simple cases such as $\frac{k}{s}$ and $(\frac{1}{k})S$; $\frac{k}{s(s+a)}$ and $(\frac{1}{k})\frac{S(s+a)}{}$; or $\frac{k}{s(s+a)(s+b)}$ and $(\frac{1}{k})\frac{s(s+a)(s+b)}{}$. Since only the angle criterion is of interest the resulting 360° shift of the angle on the root locus makes no practical difference. The fact that the gain variable is a direct relation in one case and a reciprocal relation in the corresponding inverted case maintains the "motion" of the roots with increasing gain from "pole to zero". A closed loop root is located in exactly the same place regardless of the form of the open loop polynomial ratio used

in plotting the root locus.

Based on the above statements Equations 4.26 and 4.27 may be rewritten in a more useful form, i.e.

4.26a $$\left(\frac{1}{P_1}\right) \frac{S[(S+P_1)\cdots(S+P_n)]}{[(S+\pi_1)\cdots(S+\pi_n)]} = -1$$

and

4.27a $$\left(\frac{1}{P_x}\right) \frac{S[\text{POLES OF THE UNCOMPENSATED SYSTEM}]}{[\text{ROOTS OF THE UNCOMPENSATED SYSTEM}]} = -1$$

This form of the equation allows a direct comparison of the form of Equation 4.19 and 4.27a. (4.19 being the equation for finding the lead network limit locus.)

4.19 $$\left(\frac{k_u}{Z_\ell}\right) \frac{S[\text{ZEROS OF THE UNCOMPENSATED SYSTEM}]}{[\text{ROOTS OF THE UNCOMPENSATED SYSTEM}]} = e^{\pm jm\pi}$$

For purposes of application to problems it is easier to remember the two equations in this form. Another reason for using this form is to set up solutions on the ESIAC computer. Since only the "zeros" change between Equation 4.19 and 4.27a, the number of probes in the ESIAC that must be reset is kept to a minimum.

5. CONTINUITY OF LEAD AND LAG NETWORK LIMIT LOCI AT FOCAL ROOT

The continuity of the lead and lag limit loci at their common origin on one of the uncompensated system roots, called a focal root, is of interest in sketching in the loci. Generally the lead limit locus will be determined carefully and its angle of emergence calculated graphically. The lag limit loci can be defined quickly as a continuation of the lead limit loci "through" the root, the terminating pole of locus being kept in mind. Figure 2-2 and Figure 5-1 serve to illustrate the problem.

Only the angle criterion need be considered. The usual procedure in evaluating the angle of emergence at a singularity of a function in the "Root locus form" is to construct a circle of vanishingly small radius about the singularity in question. The total phase angle of the function is then evaluated "on the circle", leaving out the angle associated with the singular point inclosed by the small circle. The total angle thus evaluated can be interpreted as equal to the angle that must be contributed by the enclosed singularity subtracted from 180°. (The sign convention being used must be observed carefully. In this discussion the 180° will cancel out). Using the angles defined in Figure 5-1 equations for the angle of emergence for the single section lead and lag limit loci will be written in an abbreviated form. Equations 4.18 and 4.26a will be used.

4.18 $$\left(\frac{K_u}{Z_\lambda}\right)\frac{(s)\left[(s+z_1)(s+z_2)\cdots\right]}{\left[(s+r_1)(s+r_2)\cdots(s+r_j)\right]} = -1$$

The angle of emergence of the limit loci defined by 4.18 at the point $-r_1$ (focal root) can be symbolized in the following manner:

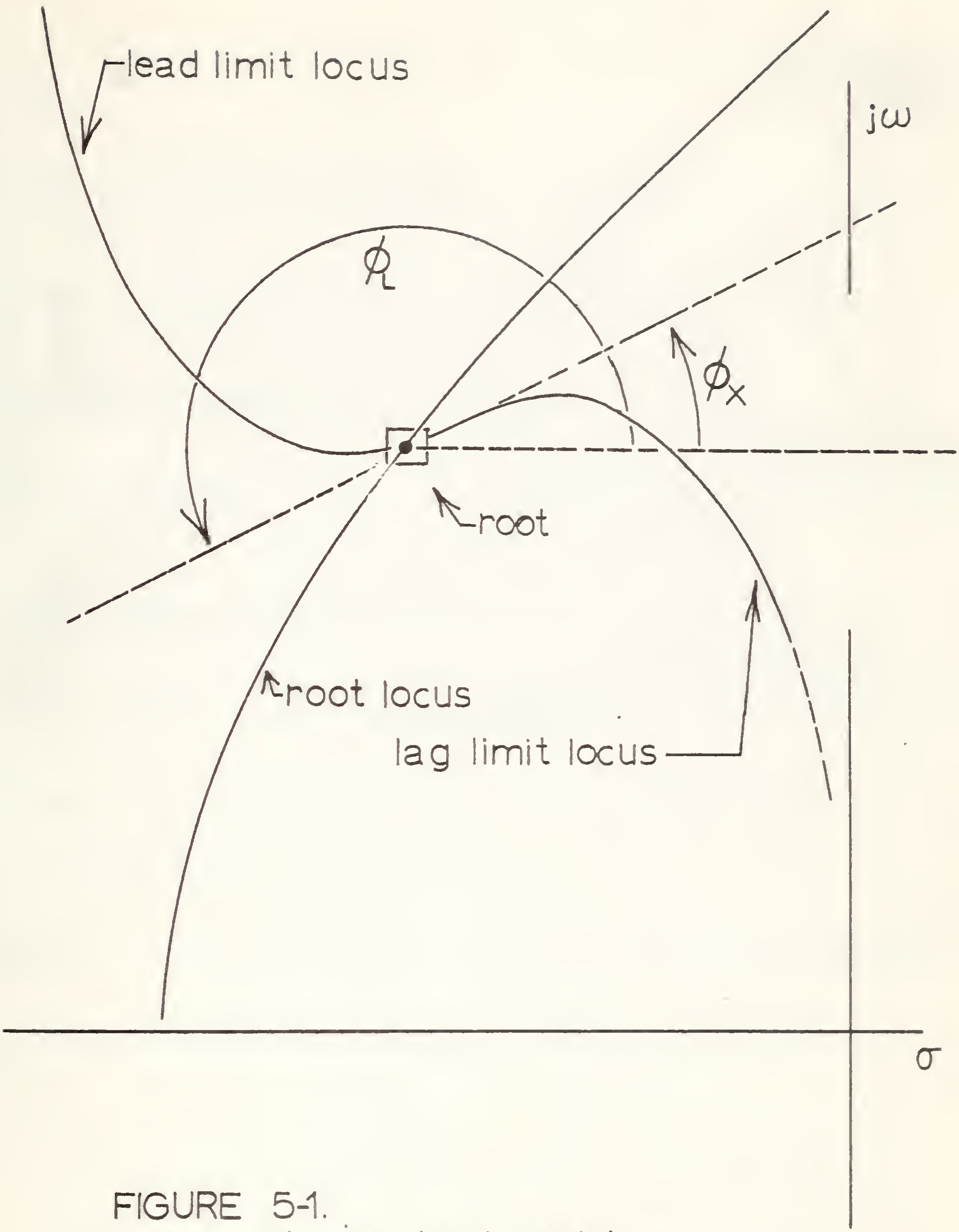


FIGURE 5-1.
continuity of lead and lag limit loci.

5.1 $$\theta_{\ell} \pm 180^{\circ} = \underline{/+r_1} + \underline{/(r_1+z_1)(r_1+z_2)\cdots} - \underline{/(r_1+r_2)(r_1+r_3)\cdots}$$

Similarly for the lag network limit loci defined by:

4.26a $$\left(\frac{1}{P_x}\right)\frac{S\left[(S+P_1)\cdots(S+P_j)\right]}{\left[(S+r_1)\cdots(S+r_j)\right]} = -1$$

The angle of emergence can be symbolized as:

5.2 $$\theta_{x} \pm 180^{\circ} = \underline{/r_1} + \underline{/(r_1+P_1)(r_1+P_2)\cdots} - \underline{/(r_1+r_2)(r_1+r_3)\cdots}$$

Now, subtract 5.2 from 5.1 to produce:

5.3 $$\theta_{\ell} - \theta_{x} = \underline{/(r_1+z_1)(r_1+z_2)} - \underline{/(r_1+P_1)(r_1+P_2)\cdots}$$

However, $-r_1$ is a point on the $\pm$ 180 root locus of $G_{ou}(s)$.

Therefore, it can be stated that:

5.4 $$\underline{/(r_1+z_1)(r_1+z_2)\cdots} - \underline{/(r_1+P_1)(r_1+P_2)\cdots} = \pm 180^{\circ}$$

Combining this result with Equation 5.3 it can be stated that

5.5 $$\theta_{\ell} - \theta_{x} = \pm 180^{\circ}$$

And therefore since the angles of emergence from a common point differ by 180° we observe that the lines have a common tangent at the point and are in fact continuous. By inspection of the general defining function for multiple section limit loci (6.3_L, 6.3_x)* it is seen that this concept applies to limit loci of any order.

*(Section 6)

6. ROOT RELOCATION ZONES: EXTENSION OF LIMIT LOCI DEFINING FUNCTIONS TO ANY NUMBER OF IDENTICAL CASCADED NETWORKS

The method of defining the root relocation zone limit lines as a limiting case of the compensated system root locus has been discussed in detail for a single section lead or lag network. The results were easily interpreted as root loci arising from pole zero configurations involving the open loop and closed loop poles and zeros. The concept can be extended to any multiplicity of identical compensator sections very simply. The extension of methods for actually plotting the locus of these higher order limit lines is not so easy. As will be shown below the difficulty lies in that for $\eta \geq 2$ only one point at a time on the limit locus can be established by root locus techniques. To find five points on the limit locus will require 5 separate root locus plots, each one defining a single point on the desired limit locus.

The basic equation is arrived at simply by considering the generalization of the argument leading to Equations 4.12 and 4.21. (The following equations will be written in pairs, L denoting lead network, x denoting lag network.) An intermediate step would be to write the compensated, open loop function for multiplicity η (see Equation 4.10) as:

$$6.1_L \qquad k_c G_{oc}(S) = \left(\frac{P_L}{Z_L} \frac{S + Z_L}{S + P_L} \right)^{\eta} k_u G_{ou}(S)$$

for a multiple lead network, and for a multiple lag network

$$6.1_x \qquad k_c G_{oc}(S) = \left(\frac{P_x}{Z_x} \frac{S + Z_x}{S + P_x} \right)^{\eta} k_u G_{ou}(S)$$

The next step is to consider these functions as $P_L \longrightarrow \infty$ and

Z_L respectively; which leads to the forms:

$$6.2_L \quad \lim_{P_l \to \infty} k_c G_{oc}(S) = \lim_{P_l \to \infty} \frac{(S+Z_l)^{\eta} k_u G_{ou}(S)}{Z_l^{\eta}\left(\frac{S^{\eta}}{P_l^{\eta}} + \frac{a_1 S^{\eta-1}}{P_l^{\eta-1}} + \cdots + \frac{a_{n-1} S}{P_l} + 1\right)}$$

$$6.2_x \quad \lim_{Z_x \to \infty} k_c G_{oc}(S) = \lim_{Z_x \to \infty} \frac{P_x^{\eta}\left(\frac{S^{\eta}}{Z_x^{\eta}} + \cdots + \frac{b_{n-1} S}{Z_x} + 1\right) k_u G_{ou}(S)}{(S+P_x)^{\eta}}$$

Which reduce to the limiting functions:

$$6.3_L \quad \lim_{P_l \to \infty} k_c G_{oc}(S) = \frac{(S+Z_l)^{\eta}}{Z_l^{\eta}} k_u G_{ou}(S)$$

$$6.3_x \quad \lim_{Z_x \to \infty} k_c G_{oc}(S) = \frac{P_x^{\eta}}{(S+P_x)^{\eta}} k_u G_{ou}(S)$$

Theoretically this is as far as the discussion need be taken. All that is necessary to find points on the limit locus corresponding to a fixed value of η is to interpret properly the root loci of the characteristic equations of 6.3, and 6.3x with Z_L and P_x considered

as the respective variable parameters. This technique will be discussed for $\eta = 2$ and $\eta = 3$. (At the conclusion of that discussion it may be clearer what is meant by the phrase "interpret properly.")

7. ROOT RELOCATION ZONES: DERIVATION OF DEFINING FUNCTIONS FOR TWO SECTION LIMIT LOCI

Consider Equation 6.3_L with $\eta = 2$.

$$7.1 \quad \lim_{P_\ell \to \infty} k_c G_{oc}(s) = \frac{(s+Z_\ell^2) k_u G_{oc}(s)}{Z_\ell^2} = \frac{(S^2 + 2SZ_\ell + Z_\ell^2) k_u [(s+Z_1)\cdots(S+Z_1)]}{Z_\ell^2 [(S+P_1)\cdots(S+P_j)]}$$

The next step is to factor the characteristic equation of 7.4 in a manner similar to that used in Equations 4.16 and 4.17. The characteristic equation can be written as:

$$7.2 \quad (S^2 + 2SZ_\ell + Z_\ell^2) k_u [(s+Z_1)\cdots(S+Z_1)] + Z_\ell^2 [(S+P_1)\cdots(S+P_j)] = 0$$

And Equation 7.2 can be regrouped as:

$$7.3 \quad (S^2 + 2SZ_\ell)[k_u(s+Z_1)\cdots(S+Z_1)] + Z_\ell^2 [k_u(s+Z_1)\cdots(S+Z_1) + (S+P_1)\cdots(S+P_j)] = 0$$

And as before it is recognized that the second bracket term factors into the roots of the uncompensated system. Or:

$$7.4 \quad k_u S(S+2Z_\ell)[(S+Z_\ell)\cdots(S+Z_1) + Z_\ell^2 [(S+\Pi_1)\cdots(S+\Pi_j)] = 0$$

And we can now put this in the standard root locus form suitable for determination of points on the "two section limit locus." (Equation 7.5)

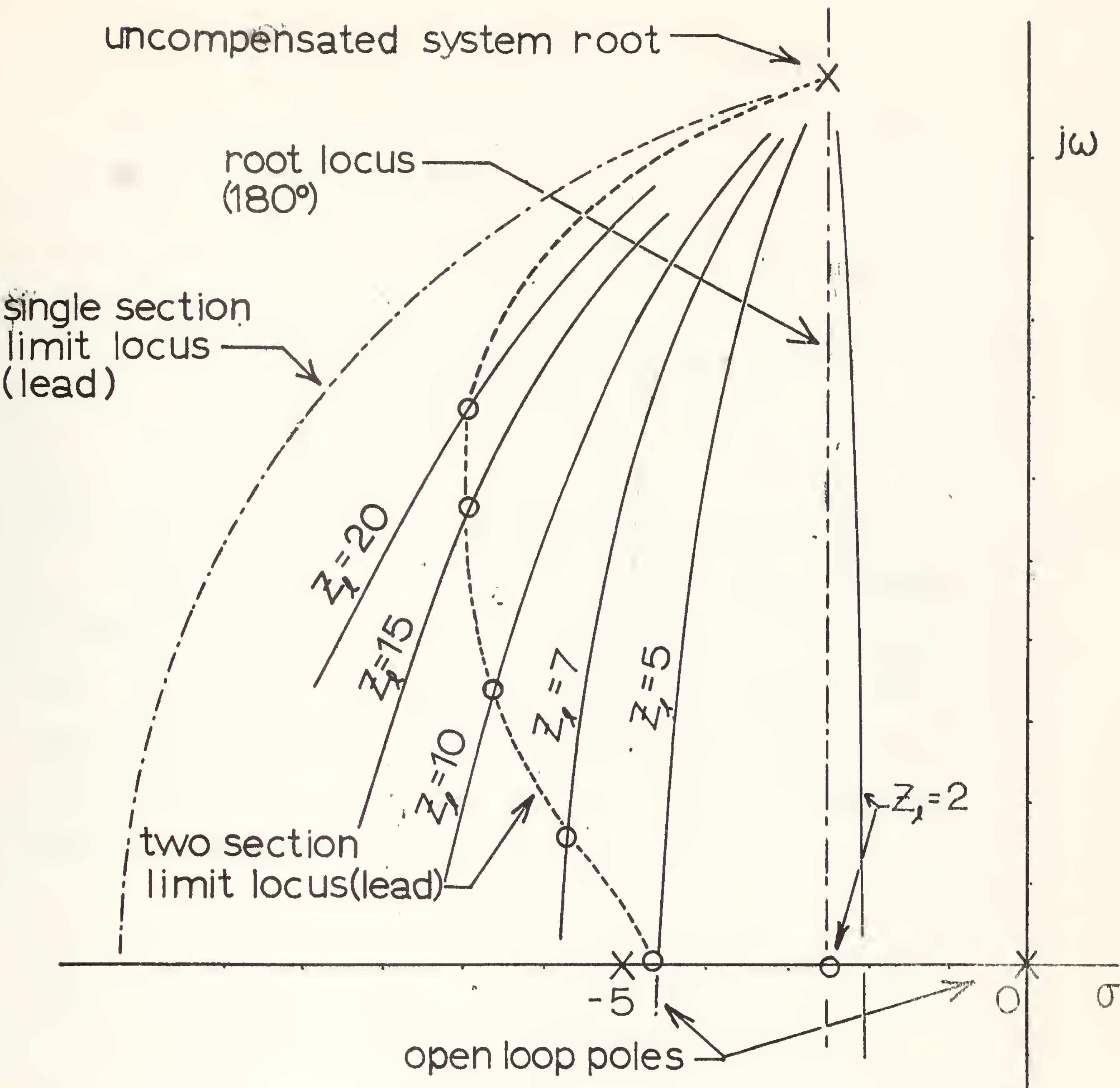
7.5 $$\left(\frac{k_u}{Z_\ell^2}\right)\frac{S(S+2Z_\ell)[(S+Z_1)\cdots(S+Z_i)]}{[(S+r_1)\cdots(S+r_j)]}\,e^{\pm jm\pi}=-1$$

Equation 7.8 should be compared with Equation 4.18 which is reproduced here: (Single section lead network)

4.18 $$\left(\frac{k_u}{Z_\ell}\right)\frac{S[(S+Z_1)\cdots(S+Z_\ell)]}{[(S+r_1)\cdots(S+r_j)]}=-1$$

The differences lie in the fact that the gain factor now involves $\frac{1}{Z_\ell^2}$, which is no real problem; but in addition to the added zero at the origin we have introduced a variable zero represented by the factor $(S+2Z_\ell)$. So we can see that as Z_L increases in magnitude the roots move in toward the poles (at the r_j) but also the root locus configuration is changed as the zero at $s = -2Z_\ell$ moves to the left along the real axis. The technique here is to pick a value for Z_L, say Z_{L1}; draw the corresponding root locus segment of interest using the angle criterion or the system defined by Equation 7.8; then locate the point on the locus corresponding to a gain of $\frac{k_u}{Z_{\ell 1}^2}$.

This is a point on the desired limit locus for $\eta = 2 \quad Z_\ell = Z_{\ell 1}$. This procedure can then be repeated until enough points are found to adequately define the limit locus in the region of interest. In Figure 7-1 this procedure has been carried out for a second order system using a spirule. The values of Z_L for each root locus segment are indicated on the diagram. This particular figure is of



SECOND ORDER SYSTEM ζ=.222 k_u=127.25

GENERALIZED ROOT LOCUS METHOD FOR DETERMINATION OF POINTS ON THE LIMIT LOCUS FOR ROOT RELOCATION ZONE ASSOCIATED WITH TWO IDENTICAL CASCADED LEAD NETWORKS. LOCUS SEGAMENTS FOR VARIOUS VALUES OF Z_ℓ DEFINED BY THE RELATION:

$$\left(\frac{k_u}{Z_\ell^2}\right)\frac{S(S+2Z_\ell)}{(Z+2.5-j11)(S+2.5+j11)} = e^{\pm jm\pi}$$

FIGURE 7-1

interest in that it is a representative second order system.

The equation defining the two section limit locus lag networks (corresponding to Equation 7.8) can be worked out as:

$$7.6 \qquad \frac{P_x^2 \, [(s+\pi_1)\cdots(s+\pi_j)]}{s\,(s+2P_x)\,[(s+P_1)\cdots(s+P_j)]} = -1$$

This function defines points on the 2 section lag limit locus.

Equation 7.6 was put in this particular form so as to correspond to Equation 4.26 for the single section lag network. It now appears that a better form or the equation is that used in Equation 4.26a. The reasons for preferring this form have been stated earlier; the use of an ESIAC computer providing the basic motive. For comparison the two equations are reproduced here:

$$4.26a \qquad \left(\frac{1}{P_x}\right) \frac{s\,[(s+P_1)\cdots(s+P_j)]}{[(s+\pi_1)\cdots(s+\pi_j)]} = -1$$

Single section lag network

$$7.6a \qquad \left(\frac{1}{P_x^2}\right) \frac{s\,(s+2P_x)\,[(s+P_1)\cdots(s+P_j)]}{[(s+\pi_1)\cdots(s+\pi_j)]} = -1$$

Two section lag network

8. ROOT RELOCATION ZONES: DERIVATION OF DEFINING FUNCTION FOR THREE SECTION LIMIT LOCI

Three identical cascaded lead networks will be considered. As before, the lag network case is a direct analogy and will not be worked out explicitly.

The limiting root locus configuration to define the three section limit locus is found by substituting $m = 3$ in Equation 6.3_L and expanding.

$$8.1 \quad \lim_{P_\ell \to \infty} k_c G_{cc}(s) = \frac{[S^3 + 3S^2 Z_\ell + 3S Z_\ell^2 + Z_\ell^3] k_u G_{cu}(s)}{Z_\ell^3}$$

In the usual manner the characteristic equation of 8.1 will be factored by the root locus technique. The roots of the uncompensated system will be singled out as a group of factors. The characteristic equation is:

$$8.2 \quad k_u [S^3 + 3S^2 Z_\ell + 3S Z_\ell^2 + Z_\ell^3][(S+Z_1)\cdots(S+Z_i)] + Z_\ell^3[(S+P_1)\cdots(S+P_j)] = 0$$

And Equation 8.2 can be written as

$$8.3 \quad k_u [S^3 + 3S^2 Z_\ell + 3S Z_\ell^2][(S+Z_1)\cdots(S+Z_i)] + Z_\ell^3[(S+P_1)\cdots(S+P_j)] = 0$$

And the corresponding root locus form is

$$8.4 \quad \left(\frac{K_u}{Z_\ell^3}\right) \frac{S\left[S^2+3SZ_\ell+3Z_\ell^2\right]\left[\text{ZEROS OF THE UNCOMPENSATED SYSTEM}\right]}{\left[\text{ROOTS OF THE UNCOMPENSATED SYSTEM}\right]} = -1$$

Though a bit involved it is still within reason to carry out the graphical solution for points on the limit locus by use of an aid such as the ESIAC. It must be observed that for each value of Z_L chosen, the quadratic term must be solved for the two additional zeros. An examination of these factors is of interest.

The equation;

$$8.5 \quad S^2 + 3SZ_\ell + 3Z_\ell^2 = 0$$

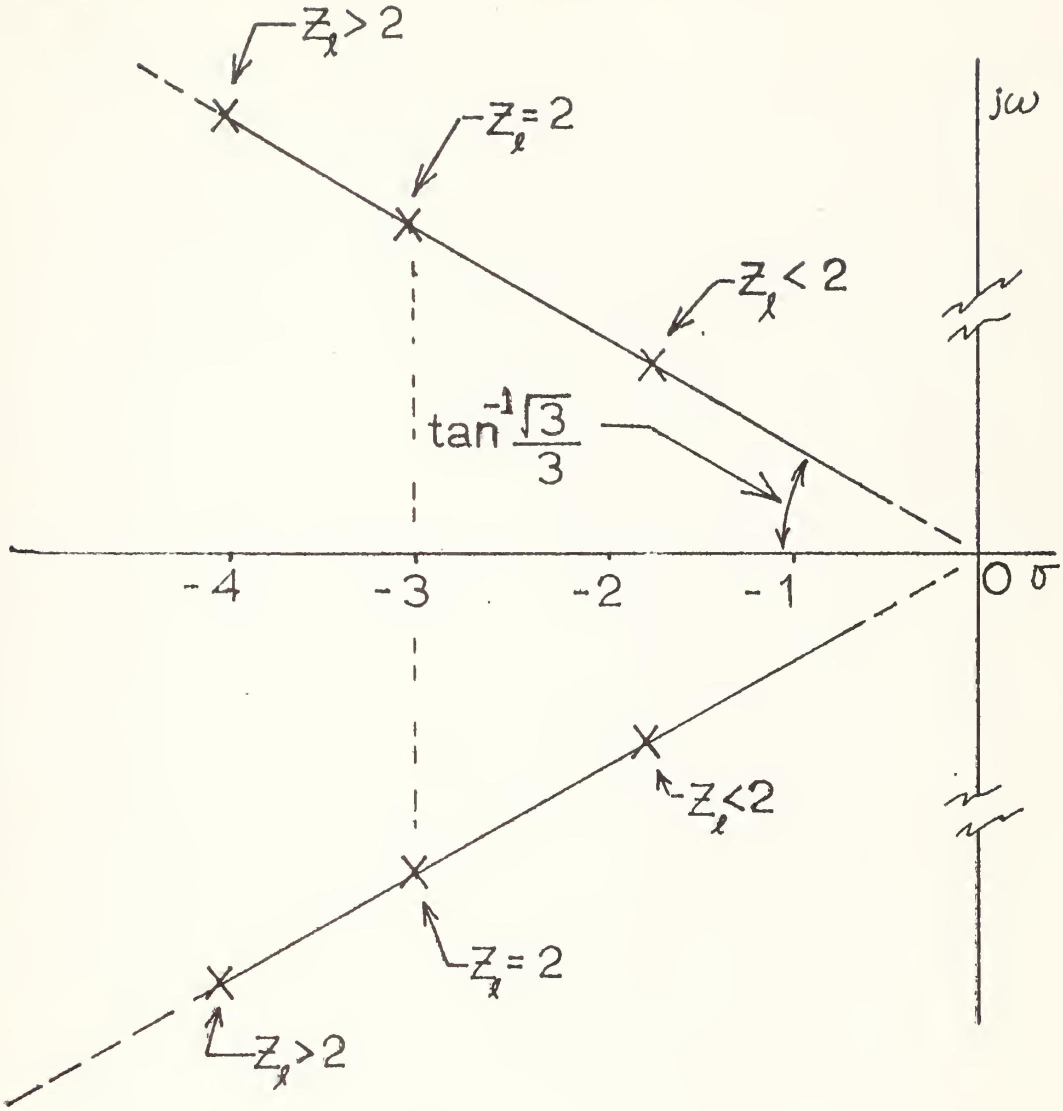
may be factored by application of the quadratic equation as

$$8.6 \quad \left[S + \frac{Z_\ell}{2}\left(3-j\sqrt{3}\right)\right]\left[S + \frac{Z_\ell}{2}\left(3+j\sqrt{3}\right)\right]$$

Which means the factor $\frac{Z_L}{2}$; which is the variable when the limit locus is being considered, merely serves to move the added zeros out along a "constant ζ line", as shown in Figure 8-1 below. This observation can facilitate a graphical solution.

<u>Conclusion</u>: (Sections 6, 7, 8)

The general root locus method for finding multiple section compensator dividing loci can be extended in theory to any order. The application to point by point graphical solutions is certainly no more laborious than other analytical techniques that have been applied to



ROOTS OF THE EQUATION

$$S^2 + 3SZ_\ell + 3Z_\ell^2 = 0$$

Z_ℓ real, positive.

$S = \sigma + j\omega$

FIGURE 8-1

servomechanism design and analysis. The method shows considerable promise as an engineering design tool in situations where there is ready access to analogue or digital computing machinery capable of root locus solutions.

9. ROOT RELOCATION ZONES: A CRITERION FOR DETERMINATION OF RELOCATING ZONES BASED ON THE COMPENSATOR EQUATIONS

It was pointed out that an extension of the root locus method of constructing root relocation zone limit lines to two or more identical cascaded networks leads to considerable complication. For two sections this involves construction of a root locus for each and every point that is to be located on the limit line. A somewhat less laborious process is a trial and error approach based on the compensator equations.

In the derivation of the relocation zone limit locus it was postulated that to place a root on the limit locus would require a compensator with pole (for a lead network) at infinity and a finite zero which would satisfy the gain restriction. To place a root on the lag network limit locus would require a zero at infinity. Since the following discussion is similar for the lead or lag network, only the lead network equation will be examined. The lag network results will be stated later. Assume that a solution for a root on the limit locus has been made. Symbolically;

If $-r$ is on the limit locus then $P_L = -\infty$

An examination of Equation 4.28 applied with the above assumption will now be made:

4.28 $$P_\ell = \frac{\omega_n \sin\frac{\phi}{n}}{\sin\left(\frac{\phi}{n} + \delta\right) - \left(\frac{K_u}{G_n}\right)^{\frac{1}{n}} \sin\delta}$$

In all situations where application of the P_L equation is valid $\sin \frac{\phi}{\eta}$ is a positive number. Therefore the denominator of 4.28 must be zero if $P_L = \infty$. If the point $-r$ lies inside the relocation zone corresponding to the value of η in the equation then the denominator of 4.28 will be a positive number. If the point $-r$ lies outside of the relocation zone then the denominator of 4.28 will be a negative number. The preceding may be summarized by stating

9.1 $$\sin\left(\frac{\phi}{\eta}+\delta\right)-\left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}\sin\delta = 0$$ If $-r$ lines on the relocation zone limit locus corresponding to η.

9.1a $$\sin\left(\frac{\phi}{\eta}+\delta\right)-\left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}\sin\delta > 0$$ If $-r$ lines inside the η section relocation zone.

9.1b $$\sin\left(\frac{\phi}{\eta}+\delta\right)-\left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}\sin\delta < 0$$ If $-r$ lies outside the η section relocation zone.

Equation 9.1 can now serve as a criterion for determining the relocation zone in which a chosen point, $-r$, lies.

<u>Example</u>: A relocation point, $-r$, has been chosen

If $\eta = n$	; Equation 9.1	negative,	$-r$ lies outside the n section zone.
If $\eta = n+1$	; Equation 9.1	positive,	$-r$ lies inside the n+1 section zone

Conclusion: In this example the smallest number of identical lead network compensator networks that can be used to compensate to the point $-r$ is $n+1$.

Equation 9.1 contains three variables since the uncompensated system gain k_u and the multiplicity of compensator networks η are usually specified by the statement of the problem. However, the angle to be supplied by the compensator, ϕ; the angle associated with the $-r$ vector, δ, and the uncompensated system gain at the new root location, G_{-r}, all depend on the choice of the point $-r$. (See Figures 2-3, 2-4) If it is desired to locate points on the limit locus for a given value of η Equation 9.1 is somewhat cumbersome. One approach is to search for points that satisfy Equation 9.1 along a "constant δ line," and thus remove one of the variables. One very convenient value of δ is 45 degrees as will be shown below.

Equation 9.1 can be expanded to:

9.2 $$\sin\frac{\phi}{\eta}\cos\delta + \cos\frac{\phi}{\eta}\sin\delta - \left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}\sin\delta = 0$$

Multiply Equation 9.2 by $\frac{1}{\cos\delta}$ to get;

9.3 $$\sin\frac{\phi}{\eta} + \cos\frac{\phi}{\eta}\tan\delta - \left(\frac{k_u}{G_{-r}}\right)^{\frac{1}{\eta}}\tan\delta = 0$$

And by collecting terms:

9.4 $$\sin\frac{\phi}{\eta} + \left[\cos\frac{\phi}{\eta} - \left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}\right]\tan\delta = 0$$

Equation 9.4 provides a convenient form to use in "searching" for points on a given relocation limit locus that intersect "constant lines." If in particular $\delta = 45^{\circ}$ then $\tan\delta = 1$ and

9.5 $$\sin\frac{\phi}{\eta} + \cos\frac{\phi}{\eta} = \left(\frac{k_u}{G_{-n}}\right)^{\frac{1}{\eta}}$$

Equation 9.5 can be put in another form by squaring both sides to produce:

9.6 $$\sin^2\frac{\phi}{\eta} + 2\sin\frac{\phi}{\eta}\cos\frac{\phi}{\eta} + \cos^2\frac{\phi}{\eta} = \left(\frac{k}{G}\right)^{\frac{2}{\eta}}$$

or

9.7 $$2\sin\frac{\phi}{\eta}\cos\frac{\phi}{\eta} = \left(\frac{k_u}{G_{-n}}\right)^{\frac{2}{\eta}} - 1$$

And Equation 9.7 can be further simplified to;

9.8 $$\sin\frac{2\phi}{\eta} = \left(\frac{k_u}{G_{-n}}\right)^{\frac{2}{\eta}} - 1 \quad ; \quad \delta = 45^{\circ}$$

And for various values of η .

$$\left.\begin{aligned} \eta = 1; \quad & \sin 2\phi = \left(\frac{k}{G}\right)^{2} - 1 \\ \eta = 2; \quad & \sin \phi = \frac{k}{G} - 1 \\ \eta = 3; \quad & \sin \frac{2\phi}{3} = \left(\frac{k}{G}\right)^{\frac{2}{3}} - 1 \\ \eta = 4, \quad & \sin \frac{\phi}{2} = \left(\frac{k}{G}\right)^{\frac{1}{2}} - 1 \end{aligned}\right\} \begin{aligned} \delta &= 45^\circ \\ \zeta &= .707 \end{aligned}$$

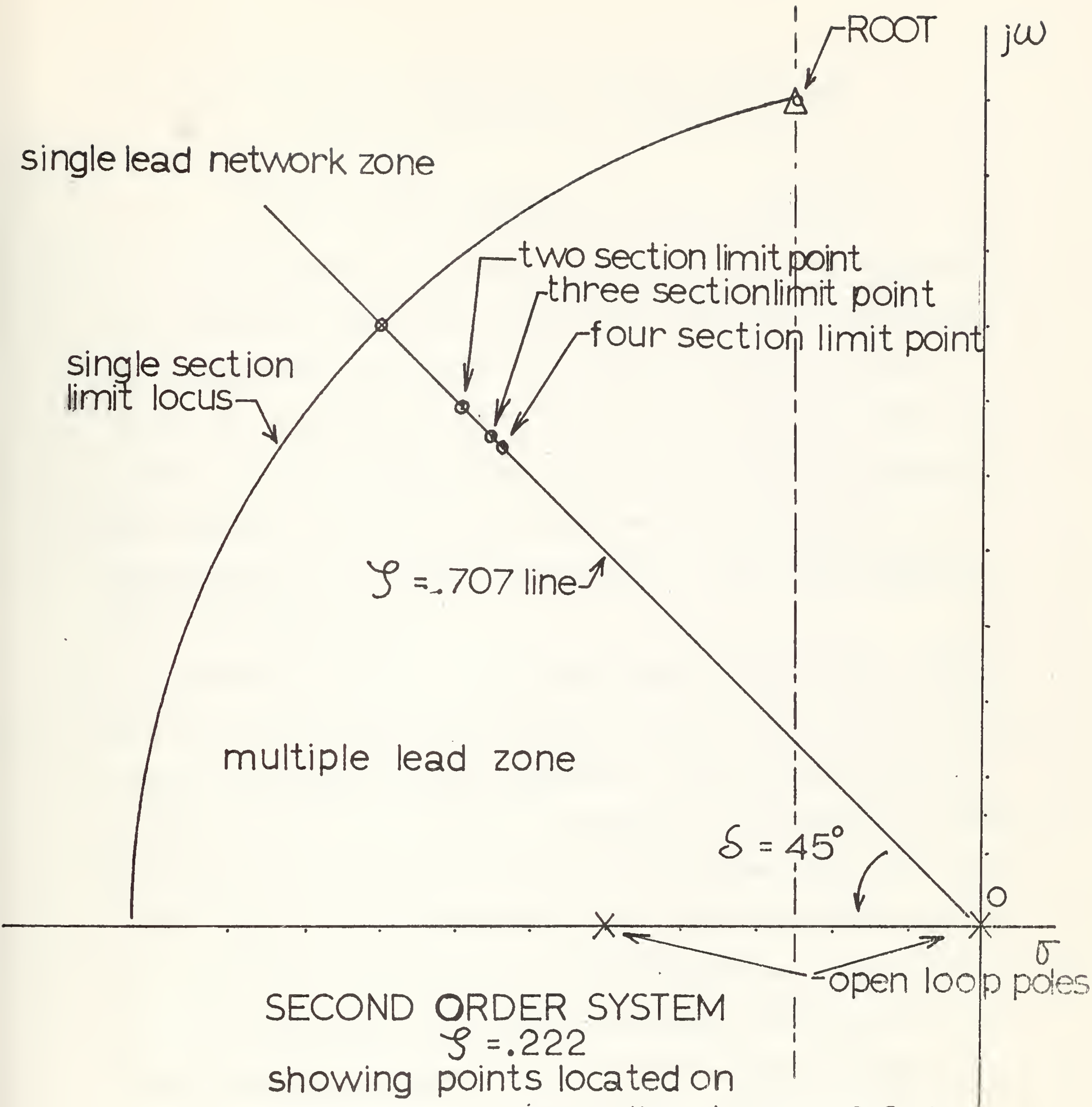
It has been observed that the single section limit locus terminates on the original uncompensated system dominant root. It can also be observed that Equation 9.1 is satisfied at the same root for any value of η . Therefore for a given value of η two points on the corresponding root relocation zone limit locus can be found easily, the first being the original root location and the second being located along the δ line with the aid of Equation 9.8. If these two points are connected by a line or arc of a circle depending on the situation a reasonable approximation to the η section limit locus has been made. Figure 9-1 shows points located along the $\delta = 45^\circ$ line for various values of η . In general as the value of η increases, the spacing between the corresponding limit loci decreases rapidly.

The equation defining points on the lag limit locus (from Equation 2.51 is)

$$9.9 \quad \sin\left(\frac{\phi}{m} + \delta\right) - \left(\frac{G_{-r}}{k_u}\right)^{\frac{1}{m}} \sin\delta = 0$$

so that it may be observed that the only change in form is the replacement of

$$\frac{k_u}{G_{-r}}\text{; in equation 9.1, by } \frac{G_{-r}}{k_u}.$$



SECOND ORDER SYSTEM
$\zeta = .222$
showing points located on
limit loci corresponding to $\eta = 1,2,3,4$.
(along $\zeta = .707$, $\delta = 45°$ line).

FIGURE 9-1

10. DIVISION OF SINGLE SECTION ROOT RELOCATION ZONES BY CONSTANT COMPENSATOR POLE ZERO RATIO LINES

(ROOT RE-LOCATION ZONES: K_v and α SPECIFIED)

In the discussion so far the key to the solution of the compensation problem has been the specification of K, the system steady state error coefficient. The fact that an infinity of solutions exist in general if K is not specified would not bother a mathematician; the engineer on the other hand would rather have a more definite answer to his questions. A solution involving as many design constraints as possible would be of interest to the engineer, even with an apparent loss of freedom of choice.

Two design constraints have actually been applied so far. First, the "error coefficient" has been chosen. Secondly, the multiplicity of compensator sections has been determined. Considering these two constraints on the solution, and if we then limit the discussion to single section filters, the possible location of system dominant roots is seen to be a well defined area of the s plane. If two constraints reduce the choice of re-location points to an area (each point of the area being related uniquely to a compensator pole/zero configuration by the basic compensator equations,) then it seems reasonable to expect the imposition of a third design constraint to limit the location points to a line in the s plane. The logical limit would be to impose a fourth constraint limiting the possible root locations to a single point.

The process of limiting solutions by increasing the number of design constraints is not compatible with the basic desire to specify the dominant root locations and then compensate the system accordingly.

However, the solutions found by the compensator equations, though mathematically correct, turn out quite often to be physically impractical. A choice of a third constraint would then normally involve the limits on the compensation network itself. The zero/pole ratio of the compensator is of primary concern. If the zero/pole spacing is small, the compensator has little effect, if the zero/pole spacing is large, the compensator has greater effect, but practical problems arise. The primary problem is one of signal attenuation in the lead network and of component size and cost in the lag network. In any event, design consideration will put a practical limit on the number, $\alpha_l = \frac{Z_l}{P_l}$ OR $\alpha_x = \frac{Z_x}{P_x}$.

As defined, the number α_l for the lead network will be smaller than unity in absolute value and α_x for the lag network will be larger than unity. In the following discussion it will be shown incidently that there is no mathematical reason for making a distinction between α_l and α_x. Since there are several physical and conceptual differences between lead and lag networks it seems wise to continue making the distinction to avoid certain pitfalls in application of the ideas.

LEAD NETWORK, K and α CONSTANT

The uncompensated system is represented as

10.1 $$k_u G_{ou}(S) = \frac{k_u (S+Z_1)\cdots(S+Z_i)}{S^n(S+P_1)\cdots(S+P_j)}$$

And design considerations dictate a fixed K, a fixed zero/pole ratio and a single section compensator network. The specified K is denoted as:

10.1a $$K_u = \lim_{S \to 0} S^n k_u G_{ou}(S) = k_u \frac{Z_1 \cdots Z_i}{P_1 \cdots P_j}$$

And the attenuation factor of the compensator is:

10.1b $$\alpha_\ell = \frac{Z_\ell}{P_\ell}$$

The compensated system is then specified as:

10.2 $$k_c G_{oc}(S) = \frac{k_c (S+Z_1) \cdots (S+Z_i)\ (S+Z_\ell)}{S^n (S+P_1) \cdots (S+P_j)\ (S+P_\ell)}$$

Where:

10.3 $$K_c = \lim_{S \to 0} S^n k_c G_{oc}(S)$$

Or

10.4 $$K_c = k_c \frac{Z_1 \cdots Z_i Z_\ell}{P_1 \cdots P_j\ P_\ell}$$

As stated above it is specified that;

10.5 $K_u = K_c$

Therefore we can solve for the compensated system gain coefficient in terms of the uncompensated system gain coefficient and the pole/zero ratio as:

10.6 $$k_u \frac{Z_1 \cdots Z_i}{P_1 \cdots P_j} = k_c \frac{Z_1 \cdots Z_i \; Z_\ell}{P_1 \cdots P_j \; P_\ell}$$

or

10.7 $$k_c = k_u \frac{P_\ell}{Z_\ell} = k_u \alpha^{-1}$$

For the lead network $k_c > k_u$.

It is now possible to write the open loop compensated system with K_v specified as:

10.9 $$k_c G_{oc}(S) = \left(\frac{P_\ell}{Z_\ell} k_u\right)\left(\frac{S + Z_\ell}{S + P_\ell}\right) G_{ou}(S)$$

In the "re-location zone limit line" investigation the next step was to take the limit of 10.09 as $P_L \longrightarrow \infty$ and then to factor out Z_L as the variable parameter in a root locus configuration. In this case, however, it will be specified that $\frac{Z_\ell}{P_\ell} = \alpha_1$; a particular finite value.

And it is observed that $P_\ell = Z_\ell \alpha_1^{-1}$. Now, with α specified, Equation 10.9 can be written in terms of α and Z_L as:

10.10 $$k_c G_{oc}(s) = (\alpha^{-1}) \frac{(S + Z_\ell)}{(S + \alpha^{-1} Z_\ell)} k_u G_{ou}(S)$$

Corresponding to the uncompensated system defined by 10.1 is the uncompensated system closed loop function

10.10a $$k_u G_{cu}(s) = \frac{k_u (s+z_1)\cdots(s+z_i)}{(s+r_1)\cdots(s+r_j)}$$

Where the r_j are the roots of the uncompensated system's characteristic equation. 10.10a is introduced here in anticipation of the form of Equation 10.12.

Equation 10.10 can be written as

10.10b $$k_c G_{oc}(s) = \frac{\alpha^{-1} k_u (s+z_1)\cdots(s+z_i)(s+z_l)}{s^n (s+p_1)\cdots(s+p_j)(s+\alpha^{-1} z_l)}$$

And the characteristic equation of this system is;

10.11 $$s^n(s+p_1)\cdots(s+p_j)(s+\alpha^{-1}z_l) + \alpha^{-1}k_u(s+z_1)\cdots(s+z_i)(s+z_l) = 0$$

Equation 10.11 can be re-arranged as;

$$s[s^n(s+p_1)\cdots(s+p_j)] + \alpha^{-1}z_l[s^n(s+p_1)\cdots(s+p_j)]$$

10.11b

$$+ s[\alpha^{-1}k_u(s+z_1)\cdots(s+z_i)] + z_l[\alpha^{-1}k_u(s+z_1)\cdots(s+z_i)] = 0$$

And another form of Equation 10.11 is;

$$s[s^n(s+p_1)\cdots(s+p_j) + \alpha^{-1}k_u(s+z_1)\cdots(s+z_i)]$$

10.12 $$+ z_l\alpha^{-1}[s^n(s+p_1)\cdots(s+p_j) + k_u(s+z_1)\cdots(s+z_i)] = 0$$

The second "bracket term" of 10.12 factors into $\left[(s+\Pi_1)\cdots(s+\Pi_j)\right]$; the denominator of Equation 10.10a.

The first "bracket term" of 10.12 presents a different problem. It could be considered that this term represents the uncompensated system with a new gain factor $k = \alpha^{-1} k_u = k_c$.

We could then quickly find the roots of the uncompensated system with this new gain at the same time we found the roots of the uncompensated system represented in Equation 10.10a. And the modified relation can be written as

$$10.13 \quad k_c G_{ou}(s) = k_c \frac{(s+Z_1)\cdots(s+Z_i)}{s^n(s+P_1)\cdots(s+P_j)}$$

And the closed loop function in factored form is

$$10.14 \quad k_c G_{cu}(s) = k_c \frac{(s+Z_1)\cdots(s+Z_i)}{(s+\Pi_1')\cdots(s+\Pi_j')}$$

Where the Π_j' are the roots of the uncompensated system using the gain factor $k_c = \alpha^{-1} k_u$. Call these roots the "k_c roots". So now we can write 10.12 as

$$10.15 \quad s\left[(s+\Pi_1')\cdots(s+\Pi_j')\right] + Z_\ell \alpha^{-1}\left[(s+\Pi_1)\cdots(s+\Pi_j)\right] = 0$$

Equation 10.15 can be considered to define a new root locus by the relation;

$$\frac{(Z_l \alpha^{-1})[(s+\Omega_1)(s+\Omega_2)\cdots(s+\Omega_j)]}{S[(s+\Omega_1')(s+\Omega_2')\cdots(s+\Omega_j')]} = -1 \qquad (10.16)$$

It can be noted that $Z_l \alpha^{-1} = P_l$ so that we can consider either P_L or Z_L as the variable parameter with α fixed.

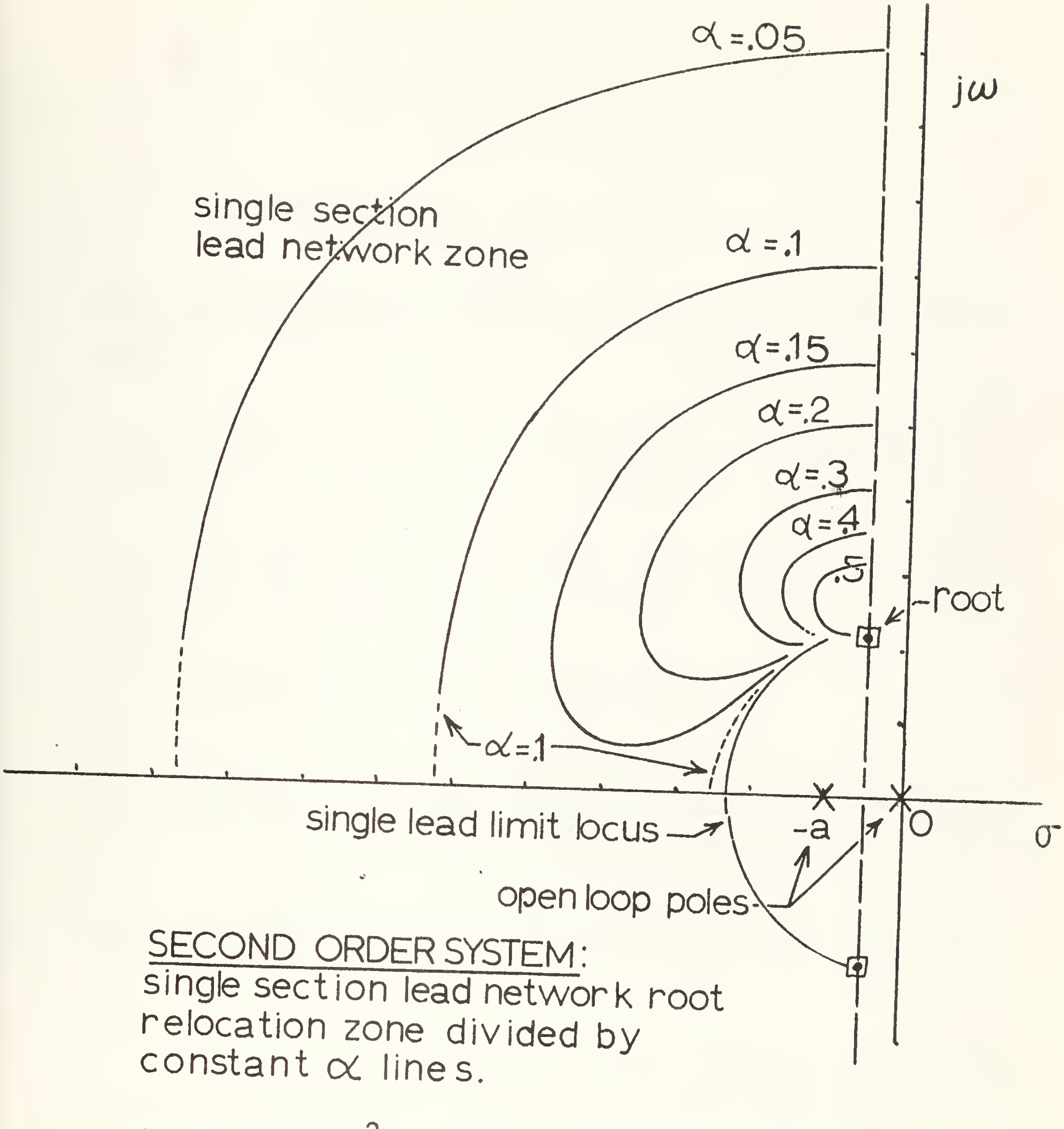
To sum up what has been done thus far the following rules for finding the locus of invarient K and α re-location, points may be listed:

To find the constant α lines, single section lead network, K specified:

1. Graph the root locus of the uncompensated system.
2. Locate all of the uncompensated system roots. Use the specified gain constant k_u. Call these the "k_u roots".
3. Multiply the uncompensated gain constant by the reciprocal of the zero/pole ratio (boost the "gain" by α^{-1} so that $k_c = \alpha^{-1} k_u$)
4. Using this new (higher) value of gain, k_c, locate a new set of roots of the uncompensated system. Call these the "k_c roots".
5. Graph a root locus which has the "k_u roots" as zeros and the "k_c roots" as poles plus an additional pole at the origin.

<u>Constant α root locus for a single section phase lag network</u>

Equation 10.1 states the form of the uncompensated system open loop function.



SECOND ORDER SYSTEM:
single section lead network root relocation zone divided by constant α lines.

$$k_u G_{ou}(s) = \frac{a^2}{4\zeta^2}\,\frac{1}{s(s+a)}$$

$\zeta = .222$

$$K_V = \frac{a}{4\zeta^2}$$

FIGURE 10-1

10.1 $$k_u G_{ou}(S) = \frac{k_{ou}(S+Z_1)\cdots(S+Z_i)}{S^n(S+P_1)\cdots(S+P_j)}$$

Where $$K_u = k_u \frac{Z_1 \cdots Z_i}{P_1 \cdots P_j}$$

The uncompensated system closed loop function written in terms of its roots is:

10.10a $$k_u G_{cu}(S) = k_u \frac{(S+Z_1)\cdots(S+Z_i)}{(S+\Pi_1)\cdots(S+\Pi_j)}$$

And as before we can call these the "k_u roots" of the uncompensated system.

The open loop compensated system function is

10.17 $$k_c G_{oc}(S) = k_c \frac{(S+Z_1)\cdots(S+Z_i)}{S^n(S+P_1)\cdots(S+P_j)} \frac{(S+Z_x)}{(S+P_x)}$$

Where $\frac{(S+Z_x)}{(S+P_x)}$ represents the lag network and $\alpha_x = \frac{Z_x}{P_x} > 1$.

Also;

10.18 $$K_c = \lim_{S\to 0} S^n k_c G_{oc}(S) = k_c \frac{Z_1 \cdots Z_i}{P_1 \cdots P_j} \frac{Z_x}{P_x}$$

The restriction the $K_c = K_u$ is imposed so that

10.19 $$k_c \frac{Z_1 \cdots Z_i}{P_1 \cdots P_j} \frac{Z_x}{P_x} = k_u \frac{Z_1 \cdots Z_i}{P_1 \cdots P_j}$$

10.20 $k_c = k_u \dfrac{P_x}{Z_x} = k_u \alpha^{-1}$ which is exactly

the result for the lead network, except that in this case $k_c < k_u$.

Going back to Equation 10.17 the substitution is now made that

$P_x = \alpha^{-1} Z_x$ where α^{-1} is considered fixed:

$$k_c G_{oc}(s) = \frac{k_c (s+Z_1)\cdots(s+Z_i)(s+Z_x)}{S^m (s+P_1)\cdots(s+P_j)(s+\alpha^{-1} Z_x)} \tag{10.21}$$

And from 10.20 Equation 10.21 can be written as;

$$k_c G_{oc}(S) = \frac{\alpha^{-1} k_u (s+Z_1)\cdots(s+Z_i)(s+Z_x)}{S^m (s+P_1)\cdots(s+P_j)(s+\alpha^{-1} Z_x)} \tag{10.22}$$

This (Equation 10.22) is of course identical in form to 10.10b.

By comparison with 10.12 we can write the characteristic equation as;

$$S\left[S^m (s+P_1)\cdots(s+P_j) + \alpha^{-1} k_u (s+Z_1)\cdots(s+Z_i)\right]$$

$$+ Z_x \alpha^{-1}\left[S^m (s+P_1)\cdots(s+P_j) + k_u (s+Z_1)\cdots(s+Z_i)\right] = 0 \tag{10.23}$$

The second "bracket term" of 10.23 factors into the "k_u roots" of the uncompensated system as before, but there is a slightly different interpretation for the first "bracket term". That is, we must find a new set of roots for the uncompensated system from the relation

$$\alpha^{-1} k_u G_{ou}(S) = \frac{\alpha^{-1} k_u (s+Z_1)\cdots(s+Z_i)}{S^m (s+P_1)\cdots(s+P_j)} = k_c \frac{(s+Z_1)\cdots(s+Z_i)}{S^m (s+P_1)\cdots(s+P_j)}$$

And the result is exactly the form of 10.14.

$$k_c G_{cu}(S) = k_c \frac{(S+Z_1)\cdots(S+Z_i)}{(S+r_1')(S+r_2')\cdots(S+r_j')} \tag{10.14}$$

Call the r_j' the "k_c roots" of the uncompensated system. The interpretation now is that these roots are "inside" the original un-compensated system; or since the system gain factor is being decreased by α^{-1}, the roots r_j' lie nearer to the P_j. In the limit the r_j' will co-incide with the P_j as the pole zero spacing of the compensator increases. In the lead section derivation the r_j' move out to the zeros of $k_u G_{ou}(s)$, either "infinite zeros" or finite termination zeros of the root locus segment.

In any event the locus of points that can be "uncompensated to" with a specified K and α for a single section phase lag network can be defined by the root locus specified by

$$\frac{(Z_x \alpha^{-1})[(S+r_1)\cdots(S+r_j)]}{S\,[(S+r_1')\cdots(S+r_j')]} = -1 \tag{10.24}$$

The collection of rules for graphing the locus of points at which a dominant root may be located using the compensator equations, (K and α fixed) may now be extended to both single lag and lead networks.

Rules for Finding the Constant α Lines
Single Section, Lead or Lag Filters

1. Graph the root locus of the uncompensated system.

2. Locate all of the uncompensated system roots.

3. Multiply the uncompensated gain constant, k_u, by the reciprocal of the zero/pole ratio, α^{-1}. This will give values of $k_c > 1$ for lead networks and $k_c < 1$ or lag networks. ("Boost" or decrease the gain constant by α^{-1} so that $k_c = k_u \alpha^{-1}$.

4. Using each new value of gain, $k_c(\alpha_i)$, locate a new set of roots, call them the "$k_c(\alpha_i)$ roots" of the uncompensated system.

5. Graph a root locus, which has the "k_u roots" as zeros, and the "$k_c(\alpha_i)$ roots" as poles plus an additional pole at the origin.

If this technique is to be used in preliminary design, it will usually be necessary only to locate the roots of interest along the "dominant segment" of root locus. The constant α lines will be nearly the shape of semi-circles connecting r_j and r'_j and can be sketched in quickly. If the value of α is critical, then the entire root locus must be drawn. Figure 10-1 shows lines of constant α constructed according to this section. Figure 12-1 shows a different configuration.

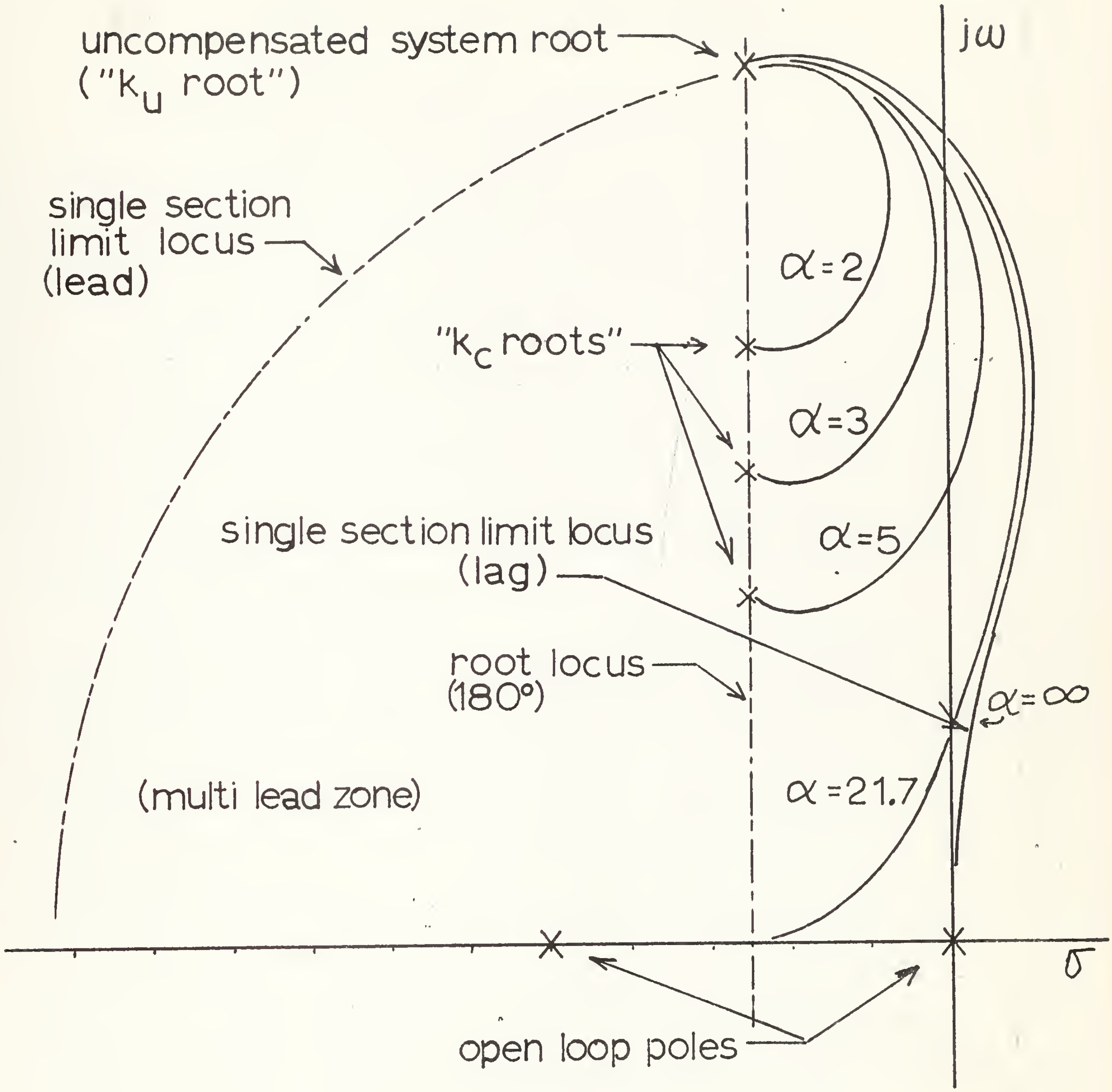
Taking the lead network as an example (see Figure 10-1). It is observed that the possibilities for choice of root re-location points for fixed K, α have been reduced to points on a well defined locus. The root re-location zone has been reduced to "practical" dimensions since all points inside the constant α line (say α_1), bounded by the uncompensated system root locus represent points for which the compensator equations will produce a compensator with $\alpha > \alpha_1$.

If for any reason the value of Z_L is now taken as a specified value, the root re-location line is reduced to a single point specified by K, η, α and Z_L. Such a restriction on Z_L is just mentioned as a possible fourth design restriction. Such a restriction would make the problem pointless and trivial to begin with, however, if an upper bound on Z_L is specified, then part of the α line is excluded, thus making it possible to avoid solutions in which Z_L is too close to the origin.

If K is allowed to vary over a range of values, then a "band" or "strip" of possible root locations is created on either side of the constant α line. In most configurations the constant α line for various K values will intersect in a point or "narrow" segment.

The root loci that have been developed in this section as "constant α lines" appear in the literature in connection with network problems. Truxal, Control System Synthesis, page 275, [5], shows how this configuration arises in connection with the transfer impedance of a reactive network terminated in a resistance. The root locus showing the effect of the loading on the reactive network roots is similar to the second order system constant α line example of this section. Evans, Control System Dynamics, page 160, [1], applies a root locus technique to the problem of selecting the capacitor to be used in a lead network. The resulting root locus configuration is exactly that presented in this section*, however the approach is quite different. Evans arranges the system so that it can be considered as a single impedance viewed from the terminals of a lead network capacitor, the capacitor being the system variable parameter.

* See Figure 12-1



SECOND ORDER SYSTEM ζ=.222 $\alpha = \frac{Z_X}{P_X}$

showing single section lag network root relocation zone divided by constant α lines.

FIGURE 10-2

11. AN ALGEBRAIC SOLUTION FOR A GENERAL SINGLE SECTION COMPENSATOR NETWORK, GAIN NOT SPECIFIED

In the paper by Ross, Warren and Thaler (Page 13 of the first draft)[1] it is pointed out that "When the gain is not restricted the compensator pole and zero may be placed at any locations which satisfy the requirement for the angle ϕ. The simplest approach in this case is to select one of the points (perhaps the zero) arbitrarily, then lay off the angle ϕ with a spirule to locate the other point (pole)". The method quoted is obviously the most rapid manual method of solution in this case, but there are at times advantages to having a mathematical rather than graphical solution at hand. Especially when there is ready access to digital computing machines.

To shorten the derivation the notation here will be abbreviated where it will not cause undue confusion. The notation $G_o(-r)$ as the evaluation of the transfer function at the point $-r$ has not been used before but is useful at times. The magnitude and angle at a point $-r$ are then respectively $|G_c(-r)|^{-1}$ and $\pm \angle G_c(-r)$.

In the notation used in the compensator equations $|G_c(-r)|^{-1} = G_{-r}$.

In the usual manner a transfer function has been specified as, $k_u G_{ou}(s)$ and it is desired to find a network with one pole and one zero that will produce a new system $\alpha^{-1} k_u G_{ou}(s) G_c(s)$ with a root at $s = -r = \sigma_r + j\omega_r$ where $\alpha = \frac{Z_c}{P_c}$ and $G_c(s) = \frac{(s+Z_c)}{(s+P_c)}$ is the transfer function of the compensator and $s = -Z_c$, $s = -P_c$ are the singular points to be included in the system. As defined above the transfer function evaluated at $-r$ is

[1] E. R. Ross, T. C Warren and G. J. Thaler, Design of Servo Compensation Based on The Root Locus Approach. AIEE Transaction Paper No. 60-779, 1960

$|G(-r)| \underline{/G(-r)}$. And considering the reciprocal of this function we have the usual gain and phase angle at a point which could be written as $G e^{j\phi_1}$.

From Figure 2-2 it can be written

11.1 $$\phi_c = \phi_{z_c} - \phi_{p_c}$$

Since -r is to be a point on the root locus of the new system the phase angle criterion may be applied in the form:

11.2 $$\underline{/G_c(-r)} + \underline{/G_o(-r)} = \pm m\pi$$

which may be written as

11.3 $$\phi_c + \phi_1 = \pm m\pi$$

By taking the tangent of both sides of Equation 11.3 it is observed that:

11.4 $$\tan\phi_c = -\tan\phi_1$$

And applying the same procedure to Equation 11.1,

11.5 $$\tan\phi_c = \frac{\tan\phi_{z_c} - \tan\phi_{p_c}}{1 + \tan\phi_{z_c}\tan\phi_{p_c}}$$

And substituting the results of 11.5 into Equation 11.4 and rearranging produces;

11.7 $$(\tan\phi_1)(1 + \tan\phi_{z_c}\tan\phi_{p_c}) = \tan\phi_{p_c} - \tan\phi_{z_c}$$

At this point an examination of Figure 2-2 indicates that

11.8 $$\tan \phi_{P_c} = \frac{\omega_n}{\sigma_n + P_c}$$

11.9 $$\tan \phi_{Z_c} = \frac{\omega_n}{\sigma_n + Z_c}$$

$\tan \phi_1$ can be evaluated as $\frac{\mathcal{I}\, G(n)}{\mathcal{R}\, G(n)}$; or the angle ϕ_1 , may be measured and $\tan \phi_1$ found by the slide rule or table.

Substitute 11.8 and 11.9 into 11.7 as follows:

11.10 $$(\tan \phi_1)\left(1 + \frac{\omega_n}{\sigma_n + Z_c}\,\frac{\omega_n}{\sigma_n + P_c}\right) = \frac{\omega_n}{\sigma_n + P_c} - \frac{\omega_n}{\sigma_n + Z_c}$$

Multiply 11.10 by $(\sigma_n + Z_c)(\sigma_n + P_c)$

11.11 $$(\sigma_n + Z_c)(\sigma_n + P_c)\tan \phi_1 + \omega_n^2 \tan \phi_1 = \omega_n(\sigma_n + Z_c) - \omega_n(\sigma_n + P_c)$$

multiply 11.11 by $(\tan \phi_1)^{-1}$

11.12 $$(\sigma_n + Z_c)(\sigma_n + P_c) + \omega_n^2 = \frac{\omega_n(\sigma_n + Z_c) - \omega_n(\sigma_n + P_c)}{\tan \phi_1}$$

and collect terms

11.13 $$Z_c P_c + (Z_c + P_c)\sigma_n + \sigma_n^2 + \omega_n^2 = \frac{\omega_n(Z_c - P_c)}{\tan \phi_1}$$

or

$$11.14 \quad Z_c P_c + (Z_c + P_c)\sigma_r + (P_c - Z_c)\frac{\omega_r}{\tan\phi_1} + (\sigma_r^2 + \omega_r^2) = 0$$

or, since $\omega_n^2 = (\sigma_r^2 + \omega_r^2)$

$$11.15 \quad Z_c P_c + (Z_c + P_c)\sigma_r + (P_c - Z_c)\frac{\omega_r}{\tan\phi_1} + \omega_n^2 = 0$$

Equation 11.15 can be solved explicitly for either P_c or Z_c in terms of the other parameters as follows:

$$11.16 \quad Z_c = -\frac{P_c\left(\sigma_r + \frac{\omega_r}{\tan\phi_1}\right) + \omega_n^2}{P_c + \sigma_r - \frac{\omega_r}{\tan\phi_1}}$$

$$11.17 \quad P_c = -\frac{Z_c\left(\sigma_r - \frac{\omega_r}{\tan\phi_1}\right) + \omega_n^2}{Z_c + \sigma_r + \frac{\omega_r}{\tan\phi_1}}$$

If the steady state error coefficient has not been specified, then it is possible to locate the root merely by picking a desired Z_c and solving 11.17 for P_c. (Or conversely, using 11.16). Once a value of Z_c or P_c has been picked, at most two real solutions for the other exist (corresponding to lead and lag networks).

It may be that for some reason α is specified. In that case the following solution can be made. Since;

11.18 $$P_c = \alpha^{-1} Z_c$$

and substituting 11.18 into 11.15:

11.19 $$Z_c^2 \alpha^{-1} + Z_c (1+\alpha^{-1}) \sigma_n + Z_c (\alpha^{-1} - 1) \frac{\omega_n}{\tan\phi_1} + \omega_n^2 = 0$$

or

11.20 $$Z_c^2 + Z_c \left[(\alpha+1) \sigma_n + (1-\alpha) \frac{\omega_n}{\tan\phi_1} \right] + \omega_n^2 \alpha = 0$$

Make the following replacements: $B = \left[(\alpha+1) \sigma_n + (1-\alpha) \frac{\omega_n}{\tan\phi_1} \right]$

$$C = \omega_n^2 \alpha$$

so that:

11.21 $$Z_c = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$ by application of the quadratic equation.

To complete the solution the value of Z_c from .21 is then substituted into .18 to find P_c.

Another interesting manipulation is to solve explicitly for in terms of Z_c.

11.22 $$Z_c^2 + Z_c \alpha \sigma_r - Z_c \alpha \frac{\omega_r}{\tan\phi_1} + Z_c\left[\sigma_r + \frac{\omega_r}{\tan\phi_1}\right] + \omega_n^2 \alpha = 0$$

or

11.23 $$\alpha\left[Z_c\left(\sigma_r - \frac{\omega_r}{\tan\phi_1}\right) + \omega_n^2\right] = \left(Z_c^2 + Z_c\left[\sigma_r + \frac{\omega_r}{\tan\phi_1}\right]\right)$$

or

11.24 $$\alpha = -\frac{Z_c^2 + Z_c D}{Z_c E + \omega_n^2} \quad \text{WHERE} \quad D = \left(\sigma_r + \frac{\omega_r}{\tan\phi_1}\right), \quad E = \left(\sigma_r - \frac{\omega_r}{\tan\phi_1}\right)$$

If a given point (-r) is chosen, it is then of interest to consider the "best" attenuation ratio that is possible at that point. Since D and E in Equation 11.24 are constants for a given choice of -r it should be possible to take the derivative of α with respect to Z_c. Setting this derivative equal to zero we can find a value of Z_c which causes α to be a maximum, and using this value in 11.24 find the value of that maximum.

11.25 $$\frac{d\alpha}{dZ_c} = -\frac{(Z_c E + \omega_n^2)(2Z_c + D) - (Z_c^2 + DZ_c)E}{(Z_c E + \omega_n^2)^2}$$

Now, set 11.25 equal to zero and solve for Z_c finding:

11.26 $$Z_{c\,\alpha_{MAX}\text{ OR MIN}} = \frac{-2W_n^2 \pm 2W_n\sqrt{W_n^2 - DE}}{2E} = \frac{-W_n^2 \pm W_n\sqrt{W_n^2 - DE}}{E}$$

Rather than put 11.26 into 11.24 it would be easier to evaluate 11.26 numerically and then substitute in 11.24 to find the maximum possible α at $-r$.

<u>Example</u>: 1 Given the system $k_u G_{ou}(s) = \frac{k}{s(s+2.6)(s+40)}$

It is desired to locate a root at: $-r = -6 + j\,10.4$.

$$W_n^2 = 144.2$$

$$\phi_1 = -244.2^\circ$$

$$\tan\phi_1 = -2.068$$

$$\phi = 64.2^\circ$$

(a) Choose $Z_c = 4$

Then from 11.17

$$P_c = -\left(\frac{4\left(-6 - \frac{10.4}{-2.068}\right) + 144.2}{4 - 6 + \frac{10.4}{-2.068}}\right) = 20.0$$

AND $$\alpha = \frac{Z_c}{P_c} = \frac{4}{20} = .2$$

Example (1) (Cont.)

(b) Choose α = .2, use Equation 20 or 21.

$$Z_c^2 + Z_c\left[(\alpha+1)\sigma_n + (1-\alpha)\frac{\omega_n}{\tan\phi_1}\right] + \omega_n^2\alpha = 0$$

$$Z_c^2 + Z_c[-11.02] + 28.8 = 0$$

$$Z_c = +7.27;\ 3.95$$

and 3.95 is the desired solution.
(Slide rule was used for all calculations).

From 11.18 $P_c = 5 Z_c = 19.75$

Which compares reasonably with Part (a) values for

P_c and Z_c.

(c) Choose Z_c = 4, calculate α from Equation 11.24.

$$D = -11.03$$

$$E = -0.97$$

$$\alpha = -\frac{Z_c^2 + Z_c(D)}{Z_c E + \omega_n^2}$$

$$\alpha = -\frac{16 + 4(-11.03)}{4(-.97) + 144.2} = -\frac{-28.12}{140.4} = .201$$

Example (1) Cont.)

(d) Find Z_c and α_{max};

$$\frac{Z_c}{\alpha_{MAX}} = \frac{-\omega_n^2 \pm \omega_n\sqrt{\omega_n^2 - DE}}{E} = -5.15, -299.2$$

And if a lead filter is under consideration:

The solution desired is $Z_c = 5.15$

Using Equation 11.24 with $Z_c = 5.15$.

$$\alpha_{MAX} = -\frac{(5.15)^2 + (5.15)(-11.03)}{(5.15)(-.97) + (144.2)} = .217$$

12. APPLICATION OF CONCEPTS OF SECTIONS 4,6,7 and 10

Typical third order system:

The example illustrated is one developed partially in the Ross-Warren thesis Page 99ff. [8] The same pole configuration was chosen to study here to permit verification of results reported by Ross and Warren. The large fold-out (Figure 12.1) should be referred to. The basic uncompensated open loop transfer function is;

$$k_u G_{ou}(s) = \frac{420}{S(S+1)(s+15)}$$

The unity feedback, closed loop roots of this system were verified (See below) to be

$$-r_3 = -16.62$$
$$-r_1 = -.31 + j5.02$$
$$-r_2 = -.31 - j5.02$$

The following functions were used in plotting the various loci on the figure.

(1) The basic 180, 360 and 540 root loci:

$$\frac{420}{S(S+1)(S+15)} = e^{\pm jm\pi}$$

(2) The single section lead network limit locus

$$\left(\frac{k_u}{Z_\ell}\right)\frac{S\,[1]}{[S+16.62][S+.31+j5.02][S+.31-j5.02]} = e^{\pm jm\pi}$$

(3) The single section lag limit locus

$$\left(\frac{1}{P_x}\right)\frac{s[s(s+1)(s+15)]}{[s+16.62][s+.31+j5.02][s+.31-j5.02]}=e^{\pm jm\pi}$$

b (4) The loci of constant compensator pole zero ratio in the single section lead zone.

$$\left(Z_\ell\alpha^{-1}\right)\frac{[(s+16.62)(s+.31+j5.02)(s+.31-j5.02)]}{s[(s+r_1')(s+r_2')(s+r_3')]}=e^{\pm jn\pi}$$

(5) The loci of constant compensator pole zero ratio in the single section lag zone.

$$\left(Z_x\alpha^{-1}\right)\frac{[(s+16.62)(s+.31+j5.02)(s+.31-j5.02)}{s[(s+r_1')(s+r_2')(s+r_3')}=e^{\pm jn\pi}$$

(6) The loci locating points on the Two section lead network limit locus.

$$\left(\frac{k_u}{Z_\ell^2}\right)\frac{s(s+2Z_\ell)}{(s+16.62)(s+.31+j5.02)(s+.31-j5.02)}=e^{\pm jm\pi}$$

(7) The loci locating points on the Two section lag network limit locus.

$$\left(\frac{1}{P_x^2}\right)\frac{s(s+2P_x)[s(s+1)(s+15)]}{(s+16.62)(s+.31+j5.02)(s+.31-j5.02)}=e^{\pm jn\pi}$$

In plotting all loci the "spirule" was used as the basic tool. However, the basic locus was checked by evaluation of the root locus equation and a plot of gain vs σ. Some of these calculations are recorded for reference.

Points on the Root locus equation were computed as follows;

σ	ω	σ	ω
0	$\pm$ 3.87	-11.0	$\pm$ 5.10
1	$\pm$ 7.07	-12	7.94
2	9.54	-13	10.30
3	11.75	-14	12.45
4	13.82	-15	14.83
5	15.81	-16	16.46
6	17.75	-17	18.38
7	19.65	-18	20.27
-.1	3.44	-19	22.14
-.2	2.95	-20	23.98
-.4	1.64		
-.5	undefined		
($\pm$ 180° locus)		($\pm$ 360° locus)	

Points of emergence were found by solving the equation:

$$\omega^2 = 3\sigma^2 + 32\sigma + 15$$

The roots of this equation are:

$$\sigma_{e_1} = -.491$$

$$\sigma_{e_2} = -10.18$$

Some of the values used to plot a curve of "Gain vs the real root" were:

σ	k
-15.5	109
-16.0	240
-16.5	383
-16.6	415.5 $\leftarrow r_3 \cong +16.62$

By a similar plot the points of emergence and entry were found on the -540° locus as:

$$\sigma_{en_{540}} = 12.1$$

$$\sigma_{em_{540}} = 21.3$$

All of the root loci configuration discussed in this thesis were verified by setting them up on the ESIAC computer. One example of an ESIAC plot is included as part of this section. See Figure 12-2. This figure is the ESIAC plot of some of the loci plotted on Figure 12.2. To facilitate comparison the numbering of the two diagrams indicate indicates the defining functions listed in this section. The small dots along the solid lines in Figure 12-2 are the marks made by the ESIAC spark recorder. The solid lines were drawn in by hand, to permit adequate photographic reproduction.

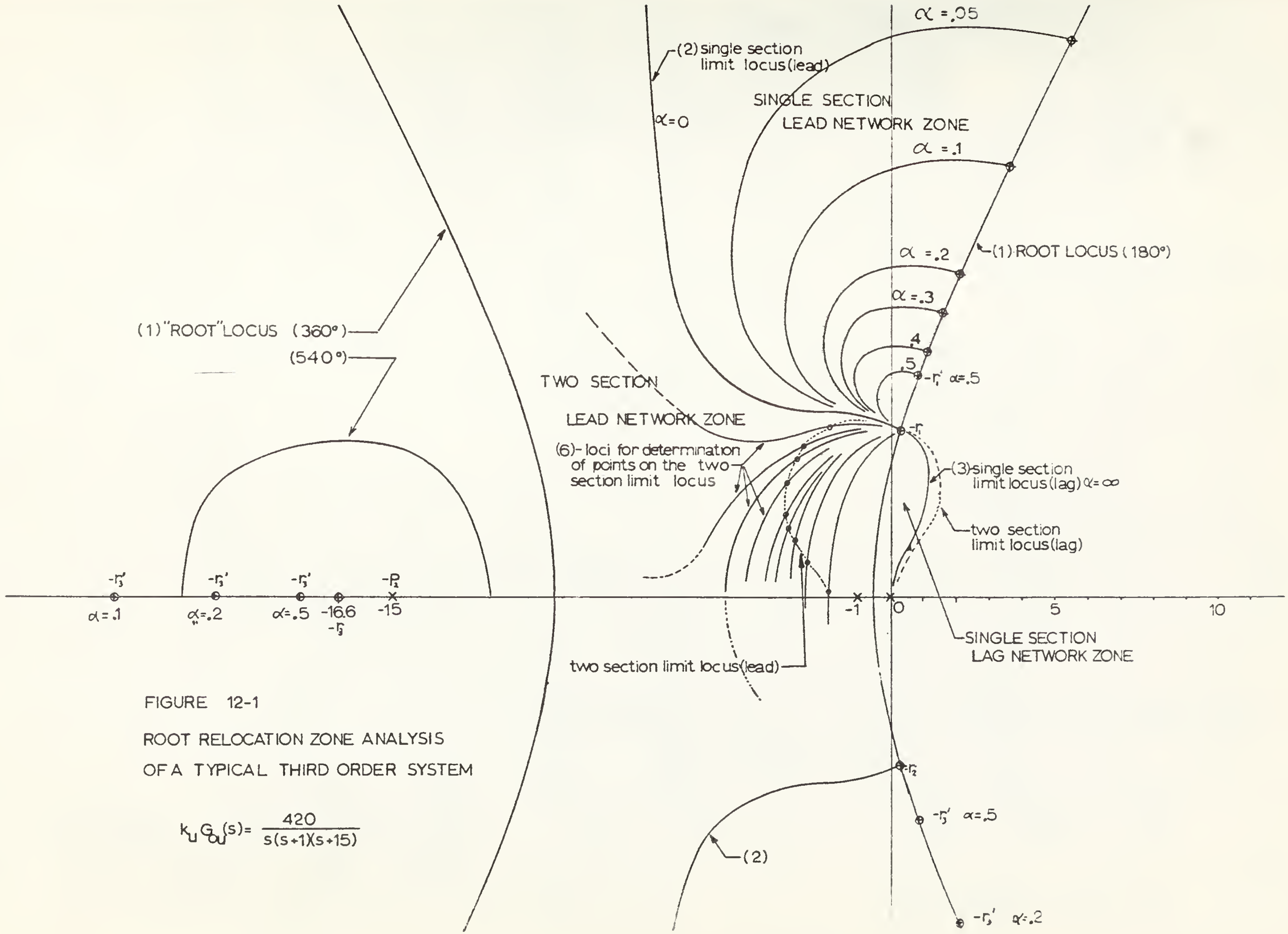


FIGURE 12-1

ROOT RELOCATION ZONE ANALYSIS OF A TYPICAL THIRD ORDER SYSTEM

$$k_u G_{ou}(s) = \frac{420}{s(s+1)(s+15)}$$

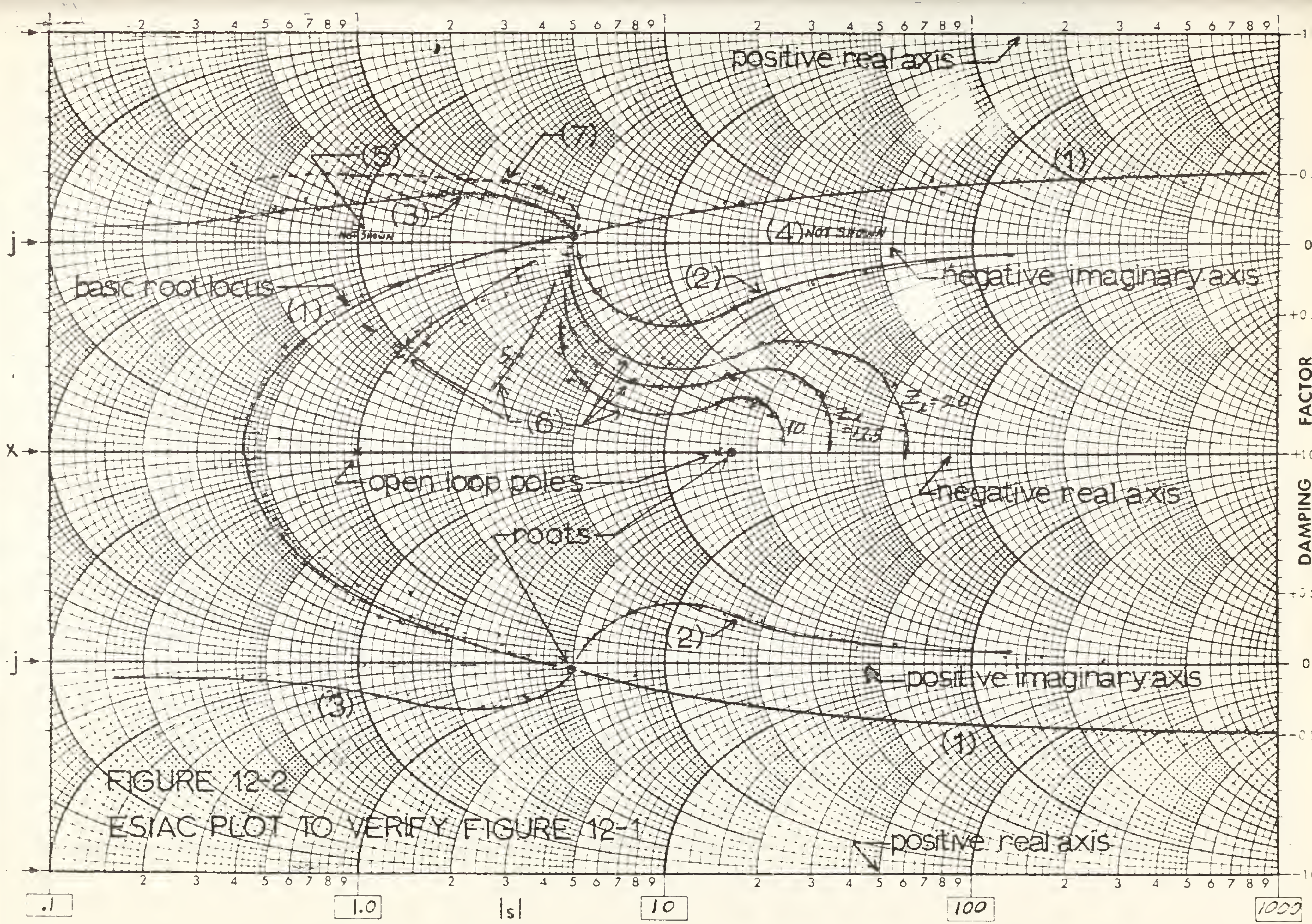


FIGURE 12-2
ESIAC PLOT TO VERIFY FIGURE 12-1

13. CONCLUSIONS AND RECOMMENDATIONS

A considerable portion of the literature on feedback control systems is devoted to the analysis and design of compensation in linear systems. The approach of this investigation falls in between the purely algebraic attack on the problem and the purely graphical method. This statement is illustrated by reference to two recently published articles.

Aseltine (7) sets out to develop a strictly linear algebraic solution for a compensator function to meet both the closed loop and open loop specifications. An examination of the examples worked by Aseltine indicate that in practice the constraints are exactly those considered here; i.e., the plant, the open loop gain, and a single pair of dominant closed loop poles are specified. The shortcomings in most situations are the same as those encountered in this thesis; there is no control over placement of roots other than the dominant pair. The algebraic solution for the compensator function becomes very cumbersome as the order of the system increases. By comparison, the graphical root locus-trigonometric solution by the compensator equations is only slightly more involved when increasingly complicated open loop functions are considered. In this thesis the root-relocation zone concept is used to test the validity of solutions, while Aseltine applies the inverse root locus technique to establish the possibility of a solution.[1]

1. J. A. Aseltine, "Feedback System Synthesis by the Inverse Root-Locus Method". IRE Conv. Rec. Part II p. 13-17, March 1956.

While the methods of this thesis are more graphical in nature than those developed by Aseltine, they do not approach the other extreme represented by a paper by Carpenter.[2] Carpenter's approach allows incorporation of all closed loop and open loop pole and zero specified locations by a fairly simple iterative graphical scheme. As in other methods, with a reasonably large number of specified poles and zeros the method rapidly becomes unmanageable. Carpenter's method has a real advantage in more easily handling complex compensator poles and zeros.

The root relocation zone concept appears to have a number of applications in analysis and design of feedback control systems. As presented here the restriction of the discussion to the idea of passive lead and lag networks and unity feedback systems may have beclouded some of the more attractive extensions of the method. Therefore, the primary conclusion (and recommendation) is that a more general development of the root relocation zone concept be made and that it be extended to a variety of situations including multiloop systems with active elements in the feed back paths.

The fundamental contribution of this pursuit may well turn out to be a better understanding of the effect of variation of parameters and addition of terms to linear functions of a complex variable of higher order. At the present it appears that in teaching compensation techniques the root-relocation zone concept leads to a better understanding of the effect of lead and lag networks than does explicit examination of the

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2. W. E. Carpenter, "Synthesis of Feed Back Systems With Specified Open-loop and Closed-loop Poles and Zeros", Space Technology Laboratories, GM-TM-0165-00355, 15 Dec. 1958.

system root locus, or analysis by the usual graphical frequency response methods. It is felt that the root locus method in general would be a great asset in the teaching of the theory of equations and functions of a complex variable. Methods such as investigated in this thesis should therefore find application in educational fields not directly linked to feedback control systems.

BIBLIOGRAPHY

1. W. R. Evans, Control-System Dynamics, McGraw-Hill Cook Company, Inc., 1954.

2. G. J. Thaler, Elements of Servomechanism Theory, McGraw-Hill Book Company, Inc., 1955.

3. G. S. Brown and D. P. Campbell, Principles of Servomechanisms, John Wiley & Sons, Inc., 1948.

4. O. J. M. Smith, Feedback Control Systems, McGraw-Hill Book Company, Inc., 1958.

5. J. G. Truxal, Automatic Feedback Control System Synthesis, McGraw-Hill Book Company, Inc., 1955.

6. L. G. Walters, Optimum Lead-Controller Synthesis in Feedback Control Systems, IRE Transactions on Circuit Theory, Vol. CT-1, pp. 45-48, March, 1959.

7. J. A. Aseltine, On the Synthesis of Feedback Systems with Open-Loop Constraints, IRE Transactions on Automatic Control, pp. 32-37, May, 1959.

8. E. R. Ross and T. C. Warren, An Exact Method of Servomechanism Compensation Developed From A Study of Phase Angle and Gain Loci, Masters Thesis, United States Naval Postgraduate School, Monterey, California, 1959.

9. ESIAC Algebraic Computer Model 10 Instruction Manual, Electro-Measurements Inc., Portland, Oregon, 1959.

10. M. L. Morgan, A New Computer for Algebraic Functions of a Complex Variable, ESI Engineering Bulletin No. 21, Electro-Measurements, Inc., Portland, Oregon, August, 1959.

11. R. V. Churchill, Introduction to Complex Variables and Applications, McGraw-Hill Book Company, Inc., 1948.

12. J. V. Uspensky, Theory of Equations, McGraw-Hill Book Company, Inc., 1948.

13. R. W. Brink, Plane Trigonometry, D. Appleton-Century Company, Inc., 1928-1940.

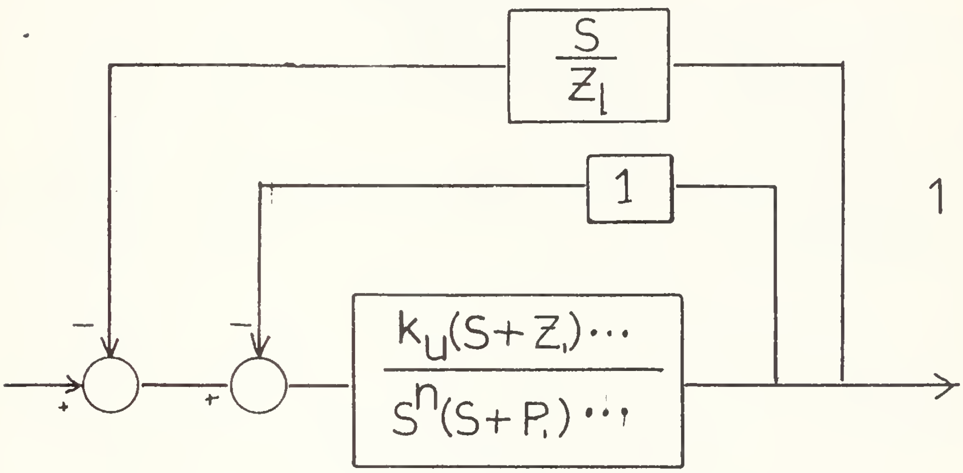
APPENDIX A

BLOCK DIAGRAM MANIPULATIONS

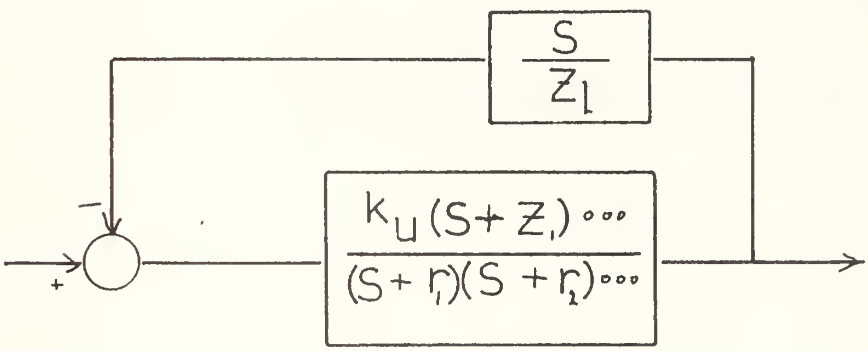
This interesting variation on the method of derivation of Limit Locus defining functions was pointed out in lecture by Dr. R. C. H. Wheeler. There are a number of ways to set up the basic block diagram. Only one representative example for lead and lag networks respectively will be presented here. Figure A-1 for a single section lead network and Figure A-2 for a single section lag network are considered elementary enough to not require further explanation. The derivations of section 4 provide the motive for setting up the diagrams.

Block diagram manipulations of this type are a standard technique for various other applications. For examples similar to those presented here see Smith, O. J. M., Feedback Control Systems, page 207, [4] and ESI Engineering Bulletin No. 21, Figure 6, [10] .

LEAD NETWORK SINGLE-SECTION LIMIT LOCUS FUNCTION, BY BLOCK DIAGRAM MANIPULATION



1

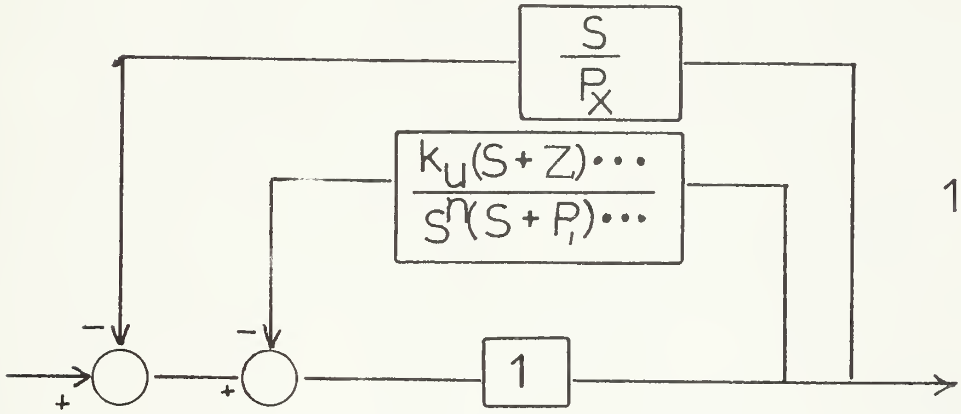


2

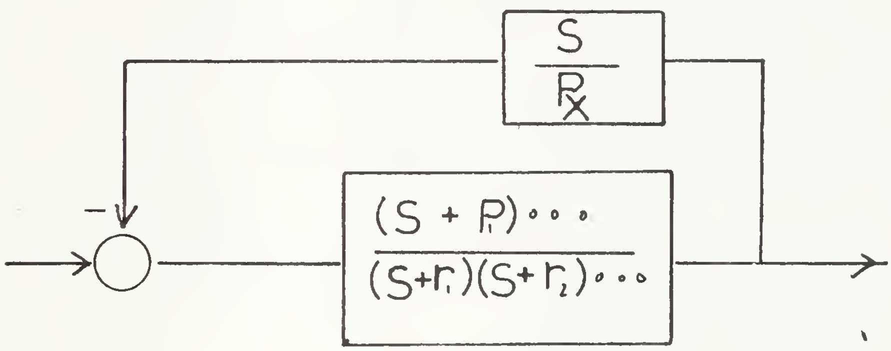
$$\frac{k_u}{Z_1}\,\frac{S(S+Z_1)\cdots}{(S+r_1)(S+r_2)\cdots} = -1 \qquad 3$$

Figure A-1

LAG NETWORK SINGLE SECTION LIMIT LOCUS FUNCTION BY BLOCK DIAGRAM MANIPULATION



1



2

$$\frac{S(S+P)(S+P)\cdots}{P_X\,(S+r_1)(S+r_2)\cdots} = -1 \qquad 3$$

Figure A-2